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THE REGULARLY VARYING DIRICHLET SERIES CONVERGING ABSOLUTELY IN HALFPLANE

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1°. *Introduction.* Let $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(n \to \infty)$, and the Dirichlet series

$$F(s) = \sum_{n=0}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1}$$

possess zero abscissa of absolute convergence. For $\sigma < 0$, we set $M(\sigma, F) = \max\{|F(\sigma + it)| : t \in \mathbb{R}\}$, and let $\mu(\sigma, F) = \max\{|a_n| \exp\{\sigma\lambda_n\} : n \geq 0\}$ be the maximal term, $\nu(\sigma, F) = \max\{n : \exp\{\sigma\lambda_n\} = \mu(\sigma, F)\}$ be the central index, and $\Lambda(\sigma, f) = \lambda_{\nu(\sigma,F)}$ be the central exponent of series (1).

As in [1], a positive measurable on $[a, 0)$ function l is said to be slowly varying if $l(c\sigma) \sim l(\sigma)$ as $\sigma \uparrow 0$ for any $c \in (0, +\infty)$, and is said to be regularly varying of index (order) $\rho \in [0, +\infty)$ if $l(\sigma) = |\sigma|^{-\rho}\alpha(\sigma)$, where α is a slowly varying function. Note that the function l is regularly varying of index ρ if and only if $l(c\sigma)/l(\sigma) \to c^{-\rho}$ as $\sigma \uparrow 0$ for any $c \in (0, +\infty)$.

The objective of this article is to find conditions upon the coefficients and exponents of series (1), under which the functions $\ln M(\sigma, F)$, $\ln \mu(\sigma, F)$, and $\Lambda(\sigma, F)$ are regularly varying.

2°. *Basic lemma.* The following lemma is basic.

Lemma 1. *If $\rho \in (0, +\infty)$, then the following assertions are equivalent*:
a) $\ln \mu(\sigma, F)$ *is a regularly varying function of index* ρ;
b) $\dfrac{|\sigma|\Lambda(\sigma, F)}{\ln \mu(\sigma, F)} \to \rho$ $(\sigma \uparrow 0)$;
c) $\dfrac{|\sigma|\Lambda(\sigma, F)}{\ln |a_{\nu(\sigma,F)}|} \to \dfrac{\rho}{\rho + 1}$ $(\sigma \uparrow 0)$;
d) $\Lambda(\sigma, F)$ *is a regularly varying function of index* $\rho + 1$.
However, if $\rho = 0$, then only assertions a), b) *and* c) *are equivalent.*

Proof. Without loss of generality we assume that $\ln \mu(-1, F) = 0$. Then (see [2], p. 182) for $\sigma \in (-1, 0)$

$$\ln \mu(\sigma, F) = \ln \mu(-1, F) + \int_{-1}^{\sigma} \Lambda(x, F)dx = \int_{-1}^{\sigma} \Lambda(x, F)dx, \tag{2}$$

and since the function $\Lambda(\sigma, F)$ does not decrease for any $c \in (0, 1)$ the inequalities

$$(1 - c)|\sigma|\Lambda(\sigma, F) \leq \ln \mu(c\sigma, F) - \ln \mu(\sigma, F) \leq (1 - c)|\sigma|\Lambda(c\sigma, F)$$

are fulfilled, therefore

$$\frac{(1 - c)|\sigma|\Lambda(\sigma, F)}{\ln \mu(\sigma, F)} \leq \frac{\ln \mu(c\sigma, F)}{\ln \mu(\sigma, F)} - 1, \quad 1 - \frac{\ln \mu(\sigma, F)}{\ln \mu(c\sigma, F)} \leq \frac{(1 - c)|\sigma|\Lambda(c\sigma, F)}{\ln \mu(c\sigma, F)},$$