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POLYADIC ANALOGS OF THE CAYLEY AND BIRKHOFF THEOREMS

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From the historical point of view, n-ary groups appeared as a generalization of the notion of a group. It seems therefore intrinsic that one of the important problems of the theory of n-ary groups is determination of analogs for the corresponding theorems in the theory of groups. In addition, the same binary result may possess several different n-ary analogs. This can be verified on the example of the Cayley theorem, whose n-ary analogs in [1] and [2] fail to coincide, though both the approaches were based on the use of ordinary, i.e., binary, permutations. Meanwhile, in the study of n-ary groups, along with the binary permutations, in [3] and [4] their n-ary analogs — finite sequences of ordinary permutations — were considered. Therefore the following problem seems to be reasonable: Obtain an n-ary analog of the Cayley theorem, where an n-ary group of n-ary permutations would play the role of the symmetric group.

The theorem proved in this article solves the mentioned problem in the most general form. In addition, in view of an implicit use of n-ary permutations in obtaining the analog of the Cayley theorem in [1], this analog is among the consequences of our theorem; moreover, every corollary of the theorem is an n-ary analog of the Cayley theorem.

Let A be an n-ary group with n-ary operation f. We define an n-ary operation g on the set A^{n-1} by analogy with the n-ary operation introduced by Post for n-ary permutations:

$$g((a'_1,\dots,a'_{n-1}),(a''_1,\dots,a''_{n-1}),\dots,(a_1^{(n)},\dots,a_{n-1}^{(n)})) =$$
$$= (f(a'_1,a''_2,\dots,a_{n-1}^{(n)},a_1^{(n)}), f(a'_2,\dots,a_{n-1}^{(n-2)},a_1^{(n-1)},a_2^{(n)}),\dots,f(a'_{n-1},a''_1,\dots,a_{n-1}^{(n)})).$$

Clearly, the Cartesian power of A^{n-1} along with the n-ary operation g is an n-ary group.

In the n-ary group A^{n-1} we select a subset $A_0 = \{(\underbrace{a,\dots,a}_{n-1}) \mid a \in A\}$. Since

$$g((\underbrace{a_1,\dots,a_1}_{n-1}),\dots,(\underbrace{a_n,\dots,a_n}_{n-1})) = (f(a_1,\dots,a_n),\dots,f(a_1,\dots,a_n)) \in A_0,$$

the set A_0 is closed with respect to g.

Proposition 1. *A_0 with the n-ary operation g is an n-ary group isomorphic to A.*

Proof. Let us define a mapping $\alpha : A \to A_0$ by the rule $\alpha : a \mapsto (\underbrace{a,\dots,a}_{n-1})$. Clearly, α is a bijection. Since

$$\alpha(f(a_1,\dots,a_n)) = (\underbrace{f(a_1,\dots,a_n),\dots,f(a_1,\dots,a_n)}_{n-1}) =$$
$$= g((\underbrace{a_1,\dots,a_1}_{n-1}),\dots,(\underbrace{a_n,\dots,a_n}_{n-1})) = g(\alpha(a_1),\dots,\alpha(a_n)),$$