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The Szegö Function on a Non-Rectifiable Arc

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Abstract—Let Γ be a simple Jordan arc in the complex plane. The Szegö function, by definition, is a holomorphic in $\mathbb{C} \setminus \Gamma$ function with a prescribed product of its boundary values on Γ. The problem of finding the Segö function in the case of piecewise smooth Γ was solved earlier. In this paper we study this problem for non-rectifiable arcs. The solution relies on properties of the Cauchy transform of certain distributions with the support on Γ.

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INTRODUCTION

Many papers published during the last decade are devoted to applications of the Riemann–Hilbert boundary-value problem in studying orthogonal polynomials and their various generalizations. These works concern both the matrix Riemann problem (the Deift–Zhou approach, e.g., [1–4]), and the usual Riemann boundary-value problem and related ones for a scalar holomorphic function defined on a plane or a Riemann surface [5–7]. We should also note the works by F. N. Garif'yanov, where the methods and notions of the theory of boundary-value problems are used for studying bi-orthogonal systems of functions (e.g., [8]). In a significant part of these studies one deals with an analytic in $\mathbb{C} \setminus \Gamma$ function $S(z)$ which has a prescribed product of its limit values from the left and from the right on the arc Γ:

$$S^+(t)S^-(t) = \rho(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{1}$$

where $a_{1,2}$ are endpoints of the arc Γ. This function is called (e.g., [6]) the Szegö function for this arc in honor of Gabor Szegö, one of the founders of the modern theory of orthogonal polynomials. A formal logarithmic transformation of this boundary condition leads to the Riemann boundary-value problem

$$\Phi^+(t) + \Phi^-(t) = f(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{2}$$

where Φ and f are logarithms of S and ρ, correspondingly. For piecewise smooth arcs the solution to this problem is known (e.g., [9], P. 442); it is obtained in terms of a Cauchy-type integral. But this technique is inapplicable for a non-rectifiable arc Γ, because the curvilinear integral $\int_\Gamma \cdot dz$ loses certainty. Thus, if we understand it in the Stieltjes sense, then the simplest sufficient (although not necessary) condition for its existence is the finiteness of the variation of z as a function defined on the arc Γ, which is equivalent to the rectifiability of this arc.

Let us mention the paper [10], where problem (1) was first solved without the requirement of the absence of zeros of the desired function. But the arc Γ in this paper is assumed to be rectifiable.

The Riemann boundary-value problem on a non-rectifiable arc was first solved in the paper [11] by the method of regularization of quasisolutions.

In this paper we solve the mentioned problem, replacing the Cauchy-type integral with the Cauchy transform of certain distributions. As a result, we weaken requirements to the arc Γ stated in [11].

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