Non-local cosmological models and localization methods

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MOTIVATION

Given the prominence of string/M-theory as a possible theory of all fundamental interactions, including gravity, the presence nonlocality within SFT is a major motivation for studying such theories.
Nonlocality is a characteristic feature of noncommutative geometry.
The covariant Witten string field theory (SFT)
E. Witten, *Noncommutative geometry and string field theory*, Nucl. Phys. B 268 (1986) 253
is an example of noncommutative geometry.
Another motivation is connected with QFT.

COSMOLOGICAL PROBLEMS

Modified gravity cosmological models have been proposed in the hope of finding to solutions to the open problems of the standard cosmological model.

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- About the early time evolution we would like to know
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  - Can we estimate influence of presence of nonlocal matter on primordial black holes formation?
  - Can we construct a realistic bounce solution without a ghost?

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- About the modern evolution epoch we would like to know
  - Why now the cosmological constant is so small?
  - Can we construct a physically acceptable dynamical dark energy (DE) model with $w < -1$?
  - Can we get a periodic crossing the $w = -1$ barrier?

To specify different types of cosmic fluids one uses a phenomenological relation between the pressure p and the energy density ϱ

$$p = w\varrho, \qquad p = E_k - V, \qquad \varrho = E_k + V$$

where w is the state parameter.
$w > 0$ — **Atoms. (4%)**
$w = 0$ — **the Cold Dark Matter. (23%)**
$w < 0$ — **the Dark Energy. (73%)**

$$w(t) = -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = -1 + \frac{2E_k}{\varrho}. \qquad (1)$$

Contemporary experiments give strong support that

$$w_{DE} \approx -1. \qquad (2)$$

We consider the case $w_{DE} < -1$. Null energy condition (NEC) is violated and there are problems of instability. A possible way to evade the instability problem for models with $w_{DE} < -1$ is to yield a phantom model as an effective one, arising from a more fundamental theory.

In particular, if we consider a model with higher derivatives such as

$$\phi e^{-\Box}\phi, \tag{3}$$

then in the simplest approximation:

$$\phi e^{-\Box}\phi \simeq \phi^2 - \phi\Box\phi, \tag{4}$$

such a model gives a kinetic term with a ghost sign. Such a possibility does appear in the string field theory framework:

I.Ya. Aref'eva, astro-ph/0410443, 2004.
I.Ya. Aref'eva and L.V. Joukovskaya, 2005;
I.Ya. Aref'eva and A.S. Koshelev, 2006;
I.Ya. Aref'eva and I.V. Volovich, 2006; 2007;
I.Ya. Aref'eva, 2007; A.S. Koshelev, 2007; I.Ya. Aref'eva and A.S. Koshelev, 2008;
I.Ya. Aref'eva, L.V. Joukovskaya, S.Yu. V. 2007; 2008;
L.V. Joukovskaya, 2007; 2008;
J.E. Lidsey, 2007;
G. Calcagni, 2006; G. Calcagni, M. Montobbio and G. Nardelli; 2007;
G. Calcagni and G. Nardelli; 2007;
N. Barnaby, T. Biswas and J.M. Cline, 2006; N. Barnaby and J.M. Cline, 2007; N. Barnaby and N. Kamran, 2007; N. Barnaby, 2008.
D.J. Mulryne and N.J. Nunes, 2008

Models with nonlocal scalar fields

We consider a model of gravity coupling with a nonlocal scalar field, which induced by strings field theory

$$S=\int d^4x\sqrt{-g}\left(\frac{m_p^2}{2}R+\frac{\xi^2}{2}\phi\,\Box\phi+\frac{1}{2}\left(\phi^2-c\,\Phi^2\right)-\Lambda'\right), \qquad (5)$$

$$\Phi=e^{\Box}\phi, \qquad (6)$$

where g is the metric,

$$\Box=\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu.$$

$m_p^2=g_4M_p^2/M_s^2$, where

M_p is a mass Planck,
$M_s=1/\sqrt{\alpha'}$ is a characteristic string scale,
g_4 is a dimensionless effective coupling constant.

$\Lambda=\frac{M_s^4}{g_4}\Lambda'$ is an effective cosmological constant.
ξ and c are positive constants.

Roots of the Characteristic Equation

Let us consider equation of motion for ϕ:

$$F(\Box)\phi = (\xi^2\Box + 1)e^{-2\Box}\phi = c\,\phi. \tag{7}$$

We assume that the metric $g_{\mu\nu}$ is given and consider (7) as an equation in ϕ.
The eigenfunctions of the Beltrami-Laplace operator

$$\Box\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi = M\phi, \tag{8}$$

also represent the solutions of equation of motion (7) with J, which is defined as a solution of the characteristic equation

$$\xi^2 J + 1 - c\,e^{2J} = 0. \tag{9}$$

The characteristic equation does not depend on metric!

Generalization

Let us consider the following nonlocal action:

$$S = \int d^4x \sqrt{-g}\alpha' \left(\frac{R}{16\pi G_N} + \frac{1}{2g_o^2}\phi\mathcal{F}(\Box)\phi - \Lambda \right) \tag{10}$$

Here G_N is the Newton constant: $8\pi G_N = 1/M_P^2$.
Function $\mathcal{F}$ is assumed to be an analytic function

$$\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n. \tag{11}$$

Equations of motion are

$$G_{\mu\nu} = \frac{8\pi G_N}{g_o^2} T_{\mu\nu} + 8\pi G_N \Lambda, \tag{12}$$

$$\mathcal{F}(\Box)\phi = 0. \tag{13}$$

In an arbitrary metric the energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\left(\partial_\mu \Box^l \phi \partial_\nu \Box^{n-1-l}\phi + \partial_\nu \Box^l \phi \partial_\mu \Box^{n-1-l}\phi - \right.$$
$$\left. - g_{\mu\nu}\left(g^{\rho\sigma}\partial_\rho \Box^l \phi \partial_\sigma \Box^{n-1-l}\phi + \Box^l \phi \Box^{n-l}\phi\right)\right)$$

(**A. Koshelev, 2007**) and can be presented in the following form:

$$T_{\mu\nu} = E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma}E_{\rho\sigma} + V\right), \qquad (14)$$

where

$$E_{\mu\nu} \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box^l \phi \partial_\nu \Box^{n-1-l}\phi, \qquad (15)$$

$$V \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \Box^l \phi \Box^{n-l}\phi. \qquad (16)$$

If $\mathcal{F}(J)$ has only simple roots J_k, then the solution of

$$\mathcal{F}(\Box)\phi = 0 \tag{17}$$

is

$$\phi = \sum_{k=1}^{N} \phi_k, \tag{18}$$

where $\Box\phi_k = J_k\phi_k$.

If we have **one simple root** ϕ_1 such that $\Box\phi_1 = J_1\phi_1$, then

$$E_{\mu\nu}(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n-1}\partial_\mu\phi_1\partial_\nu\phi_1 = \frac{\mathcal{F}'(J_1)}{2}\partial_\mu\phi_1\partial_\nu\phi_1.$$

$$V(\phi_1) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} J_1^{n}\phi_1^2 = \frac{J_1}{2}\sum_{n=1}^{\infty} f_n n J_1^{n-1}\phi_1^2 = \frac{J_1\mathcal{F}'(J_1)}{2}\phi_1^2.$$

In the case of **two simple roots** ϕ_1 and ϕ_2 we have

$$E_{\mu\nu}(\phi_1 + \phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2) + E^{cr}_{\mu\nu}(\phi_1, \phi_2), \tag{19}$$

where the cross term

$$E^{cr}_{\mu\nu}(\phi_1, \phi_2) = A_1 \partial_\mu \phi_1 \partial_\nu \phi_2 + A_2 \partial_\mu \phi_2 \partial_\nu \phi_1. \tag{20}$$

$$A_1 = \frac{1}{2} \sum_{n=1}^{\infty} f_n J_1^{n-1} \sum_{l=0}^{n-1} \left(\frac{J_2}{J_1} \right)^l = \frac{\mathcal{F}(J_1) - \mathcal{F}(J_2)}{2(J_1 - J_2)} = 0, \tag{21}$$

$$A_2 = 0. \tag{22}$$

So, the cross term $E^{cr}_{\mu\nu}(\phi_1, \phi_2) = 0$ and

$$E_{\mu\nu}(\phi_1 + \phi_2) = E_{\mu\nu}(\phi_1) + E_{\mu\nu}(\phi_2) \tag{23}$$

Similar calculations shows

$$V(\phi_1 + \phi_2) = V(\phi_1) + V(\phi_2). \tag{24}$$

In the case of N **simple roots** the following formula has been obtained:

$$T_{\mu\nu} = \sum_{k=1}^{N} \mathcal{F}'_{,J}(J_k) \left(\partial_\mu \phi_k \partial_\nu \phi_k - \frac{1}{2} g_{\mu\nu} \left(g^{\rho\sigma} \partial_\rho \phi_k \partial_\sigma \phi_k + J_k \phi_k^2 \right) \right). \quad (25)$$

Note that the last formula is exactly the energy-momentum tensor of many free massive scalar fields. If $\mathcal{F}(J)$ has simple real roots, then positive and negative values of $\mathcal{F}'_{,J}(J_i)$ alternate, so we can obtain phantom fields.

I.Ya. Aref'eva, L.V. Joukovskaya, S.V.,
J. Phys A **41** (2008) 304003, arXiv:0711.1364

Considering the following local action

$$S_{loc} = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} - \Lambda\right) + \sum_{i=1}^{N_1} S_i, \qquad (26)$$

where

$$S_i = -\frac{1}{g_o^2}\int d^4x\sqrt{-g}\frac{\mathcal{F}'(J_i)}{2}\left(g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + J_i\phi_i^2\right).$$

We can see that solutions of the Einstein equations and equations in ϕ_k obtained from this local action solves the initial nonlocal equations.

Special solutions to nonlocal equations can be found as solutions to system of local (differential) equations.
If $\mathcal{F}(J)$ has an infinity number of roots then one nonlocal model corresponds to infinity number of different local models and the initial nonlocal action (10) generates infinity number of local actions (26).

We should prove that the way of localization is self-consistent. To construct local action (26) we assume that equations

$$\Box\phi_k = J_k\phi_k$$

are satisfied.
The method of localization is correct only if these equations can be obtained from the local action S_{loc}.
Indeed,

$$\frac{\delta S_{loc}}{\delta\phi_k} = 0 \Leftrightarrow \Box_g\phi_k = J_k\phi_k.$$

We also obtain from S_{loc} the equations:

$$G_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu}(\phi) - \Lambda g_{\mu\nu} \right), \tag{27}$$

where ϕ is

$$\phi = \sum_{k=1}^{N} \phi_k, \tag{28}$$

and $T_{\mu\nu}(\phi)$ is given by (25).

There exists the following algorithm of localization

- Find roots of the function $\mathcal{F}(J)$ and calculate orders of them.
- Select an finite number of simple roots.
- Construct the corresponding local action.
- Vary the obtained local action and get a system of the Einstein equations and equations of motion.
- Seek solutions of the obtained local system.

The Ostragradski representation

- M. Ostrogradski, *Mémoire sur les équations differentielles relatives aux problèmes des isoperimétres*, Mem. St. Petersbourg VI Series, V. 4 (1850) 385–517
- A. Pais and G.E. Uhlenbeck, *On Field Theories with Nonlocalized Action*, Phys. Rev. 79 (1950) 145–165

Let $\mathcal{F}$ is a polynomial:

$$\mathcal{F}(\Box) = \mathcal{F}_1(\Box) \equiv \prod_{j=1}^{N} \left(1 + \frac{\Box}{\omega_j^2}\right), \qquad (29)$$

all roots, which are equal to $-\omega_j^2$, are simple.
We want to get the Ostrogradski representation for

$$\mathcal{L}_F = \phi \mathcal{F}_1(\Box)\phi. \qquad (30)$$

For the local Lagrangian $\mathcal{L}_F$ the Ostrogradski representation is as follows

$$\mathcal{L}_F \equiv \phi \mathcal{F}_1(\Box)\phi \cong L_I = \sum_{j=1}^{N} c_j \phi_j (\Box + \omega_j^2)\phi_j, \qquad (31)$$

where the sign $' \cong '$ means equality up to a full derivative.

$$\phi_j = \prod_{k=1, k\neq j}^{N} \left(1 + \frac{1}{\omega_k^2}\Box\right)\phi, \qquad \Rightarrow \qquad \left(\Box + \omega_j^2\right)\phi_j = 0. \qquad (32)$$

Substituting ϕ_j in L_l, one gets that (31) is equivalent to:

$$L_l = \mathcal{L}_F \qquad \Leftrightarrow \qquad \sum_{k=1}^{N} \frac{c_k \omega_k^4}{\omega_k^2 + \Box} = \frac{1}{\mathcal{F}_1(\Box)}$$

$$c_k = \frac{\mathcal{F}_1'(-\omega_k^2)}{\omega_k^4}, \qquad \mathcal{F}_1'(J) \equiv \frac{d\mathcal{F}_1}{dJ}.$$

Let $\mathcal{F}_1(\Box)$ has two real simple roots, it is evident that $\mathcal{F}_1' > 0$ in one and only one root. Therefore, we get a model with one phantom scalar field and one standard scalar field.

A modification that assumes the existence of a new dimensional parameter M_* can be of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + \frac{1}{2}R\mathcal{F}\left(\frac{\Box}{M_*^2}\right)R - \Lambda + \mathcal{L}_{\rm matter}\right], \qquad (33)$$

where M_* is the mass scale at which the higher derivative terms in the action become important, $8\pi G_N = 1/M_P^2$.
An analytic function $\mathcal{F}(\Box/M_*^2) = \sum\limits_{n\geqslant 0} f_n\Box^n$.

Biswas T., Mazumdar A., and Siegel W., 2006, *JCAP* **0603** 009 (arXiv:hep-th/0508194)
Biswas T., Koivisto T., and Mazumdar T., 2010, *JCAP* **1011** 008 (arXiv:1005.0590)
Biswas T., Koshelev A.S., Mazumdar T., Vernov S.Yu., *JCAP* **1208** (2012) 024 (arXiv:1206.6374)

By virtue of the field redefinition one can transform the non-local gravity action (35) as follows:

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}(1+\Phi)R + \frac{1}{2}\tau\mathcal{F}\left(\frac{\Box}{M_*^2}\right)\tau - \frac{M_P^2}{2}\Phi\tau - \Lambda\right) \quad (34)$$

with two new scalar fields Φ and τ.
Variation w.r.t. Φ gives $\tau = R$ and, therefore, the connection (34) with action (35) is obvious.

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + \frac{1}{2} R \mathcal{F} \left(\frac{\Box}{M_*^2} \right) R - \Lambda + \mathcal{L}_{\mathrm{M}} \right], \qquad (35)$$

where M_P is the Planck mass, Λ is the cosmological constant, M_* is the mass scale at which the higher derivative terms in the action become important.

The action of the covariant d'Alembertian on a scalar is given by

$$\Box \equiv g^{\mu\nu} D_\mu D_\nu = g^{\mu\nu} D_\mu \partial_\nu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \right),$$

where D_μ is the covariant derivative.

Introducing dimensionless coordinates $\bar{x}_\mu = M_* x_\mu$ and $\bar{M}_P = M_P / M_*$ we get $\mathcal{F}(\Box/M_*^2) = \mathcal{F}(\bar{\Box})$, where $\bar{\Box}$ is the d'Alembertian in terms of dimensionless coordinates.

We shall use dimensionless coordinates only (omitting the bars).

Variation of action (35) yields the following system:

$$\frac{1}{2}[M_P^2 + 2\mathcal{F}(\Box)R]\,(2R^\mu_\nu - \delta^\mu_\nu R) = \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\Big[g^{\mu\rho}\partial_\rho\Box^l R\partial_\nu\Box^{n-l-1}R +$$

$$+g^{\mu\rho}\partial_\nu\Box^l R\partial_\rho\Box^{n-l-1}R - \delta^\mu_\nu\left(g^{\rho\sigma}\partial_\rho\Box^l R\partial_\sigma\Box^{n-l-1}R + \Box^l R\Box^{n-l}R\right)\Big] +$$

$$+2(g^{\mu\rho}D_\rho\partial_\nu - \delta^\mu_\nu\Box)\mathcal{F}(\Box)R - \frac{1}{2}R\mathcal{F}(\Box)R\delta^\mu_\nu - \Lambda\delta^\mu_\nu + T^\mu_\nu\,,$$

where D_μ is the covariant derivative, T^μ_ν is the energy–momentum tensor of matter.
The trace equation is

$$M_P^2 R - \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\left(\partial_\mu\Box^l R\partial^\mu\Box^{n-l-1}R + 2\Box^l R\Box^{n-l}R\right) - 6\Box\mathcal{F}(\Box)R = 4\Lambda - T^\mu_\mu\,. \quad (36)$$

The following ansatz

$$\Box R = r_1 R + r_2, \tag{37}$$

with constants $r_1 \neq 0$ and r_2 is useful in finding exact solutions. Therefore,

$$\begin{aligned} \Box^n R &= r_1^n \left(R + \frac{r_2}{r_1} \right) \quad \text{for} \quad n > 0, \\ \mathcal{F}(\Box) R &= \mathcal{F}_1 R + \frac{r_2}{r_1} (\mathcal{F}_1 - f_0) \quad \text{where} \quad \mathcal{F}_1 \equiv \mathcal{F}(r_1). \end{aligned} \tag{38}$$

If the scalar curvature R satisfies (47), then we get local equations

$$\frac{1}{2}\left[M_P^2 + 2\left(\mathcal{F}(r_1)R + \frac{r_2}{r_1}(\mathcal{F}(r_1) - f_0)\right)\right](2R^\mu_\nu - \delta^\mu_\nu R) = T^\mu_\nu +$$
$$+\mathcal{F}'(r_1)\left[\partial^\mu R\partial_\nu R - \frac{\delta^\mu_\nu}{2}\left(g^{\sigma\rho}\partial_\sigma R\partial_\rho R + r_1\left(R + \frac{r_2}{r_1}\right)^2\right)\right] - \Lambda\delta^\mu_\nu +$$
$$+2\mathcal{F}(r_1)\left[D^\mu\partial_\nu R - \delta^\mu_\nu(r_1 R + r_2)\right] - \frac{\delta^\mu_\nu}{2}\left[\mathcal{F}(r_1)R^2 - \frac{r_2^2}{r_1^2}(\mathcal{F}(r_1) - f_0)\right], \tag{39}$$

where $\mathcal{F}'$ is the first derivative of $\mathcal{F}$ with respect to the argument.

We proceed to consider a traceless radiation along with a cosmological constant. Under condition (47) the trace equation with $T^\mu_\mu = 0$ becomes especially simple:

$$AR + \mathcal{F}'(r_1)\left(2r_1 R^2 + \partial_\mu R \partial^\mu R\right) + B = 0\,, \qquad (40)$$

where constants A and B are defined as follows:

$$A = 4\mathcal{F}'(r_1)r_2 - M_P^2 - 2\frac{r_2}{r_1}(\mathcal{F}(r_1) - f_0) + 6\mathcal{F}(r_1)r_1, \quad B = 4\Lambda + \frac{r_2}{r_1}M_P^2 + \frac{r_2}{r_1}A.$$

The simplest way to get a solution to equation (40) is to impose $\mathcal{F}'(r_1) = 0$ and to put $A = B = 0$.

These relations fix values of r_1, r_2 and the cosmological constant:

$$r_2 = -\frac{r_1[M_P^2 - 6\mathcal{F}(r_1)r_1]}{2[\mathcal{F}(r_1) - f_0]}, \qquad \Lambda = -\frac{r_2 M_P^2}{4r_1} = M_P^2 \frac{[M_P^2 - 6\mathcal{F}(r_1)r_1]}{8[\mathcal{F}(r_1) - f_0]}. \tag{41}$$

Under conditions $A = B = \mathcal{F}'(r_1) = 0$ the complete set of equations (39) simplifies to

$$2\mathcal{F}(r_1)(R + 3r_1)G^{\mu}_{\nu} = T^{\mu}_{\nu} + 2\mathcal{F}(r_1)\left[g^{\mu\rho}D_{\rho}\partial_{\nu}R - \frac{1}{4}\delta^{\mu}_{\nu}\left(R^2 + 4r_1R + r_2\right)\right] . \tag{42}$$

In general one is required to include radiative sources to get exact solutions of all equations.
Equations are general and we do not take into account the properties of the metric.

Relation between the anzats and R^2 modified gravity

Let us consider the $f(R)$ gravity model, described by the action:

$$S_f = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}f(R) + \mathcal{L}_{\mathrm{M}}\right). \tag{43}$$

We get the following equations:

$$M_P^2\left(f^{(1)}(R)R^\mu_\nu - \frac{1}{2}f(R)\delta^\mu_\nu - (g^{\mu\rho}D_\rho\partial_\nu - \delta^\mu_\nu\Box)f^{(1)}(R)\right) = T^\mu_\nu, \tag{44}$$

where $f^{(1)}(R)$ is the first derivative of $f(R)$ with respect to R.

From (44) for a traceless matter, we get the following trace equation:

$$f^{(1)}(R)R - 2f(R) + 3\Box f^{(1)}(R) = 0. \qquad (45)$$

One can see that at

$$f(R) = \frac{\mathcal{F}(r_1)}{M_P^2}\left[R^2 + 6r_1 R + 3r_2\right], \qquad (46)$$

equation (45) is coincide with the ansatz (47). Moreover, equations (42) are equivalent to (44). Note that condition (41) gives the following connection:

$$M_P^2 = \frac{2}{r_1}\left[3\mathcal{F}(r_1)r_1^2 - (\mathcal{F}(r_1) - f_0)r_2\right].$$

Any solution of the modified gravity model (43) with $f(R)$ given by (46) and traceless matter is a solution of the initial system of nonlocal equations on condition that $A = B = \mathcal{F}'(r_1) = 0$ and the anzats (47) is satisfied.

A.S. Koshelev, S.Yu. Vernov, Physics of Particles and Nuclei Letters 11(7): 960-963

The Hyperbolic Cosine Bounce

In the FLRW metric with the interval:

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

the ansatz

$$\Box R = r_1 R + r_2, \tag{47}$$

becomes a third order differential equation for the Hubble parameter, $H = \dot{a}/a$,

$$\dddot{H} + 7H\ddot{H} + 4\dot{H}^2 + 12H^2\dot{H} = -2r_1 H^2 - r_1\dot{H} - \frac{r_2}{6}, \tag{48}$$

where dot defines derivative w.r.t cosmic time.

The exact bounce solution is given by

$$a(t) = a_0 \cosh \sqrt{\frac{r_1}{2}} t, \quad \Rightarrow \quad H = \sqrt{\frac{r_1}{2}} \tanh \left(\sqrt{\frac{r_1}{2}} t \right) \qquad (49)$$

This solution satisfies the ansatz with the specific parameter combination

$$\Box R = r_1 R - 6 r_1^2.$$

The radiation energy density at the bounce point, ρ_0, which must be positive:

$$\rho_0 = \frac{3(M_p^2 r_1 - 2\lambda f_0 r_2)(r_2 - 12 h_1 M_*^4)}{12 r_1^2 - 4 r_2}, \qquad (50)$$

where $h_1 = \ddot{H}/M_*^3$ characterizes the acceleration of the universe at the bounce point and plays the role of an 'initial condition'.
From $r_2 = -6r_1^2$ we get

$$\Lambda = \frac{3}{2} r_1 M_P^2, \qquad \rho_0 = -\frac{27}{2} \lambda \mathcal{F}_1 r_1^2. \qquad (51)$$

The radiation is non-ghost like, provided

$$\mathcal{F}(r_1) < 0. \qquad (52)$$

GENERALIZATION

Alexey S. Koshelev, 2013.

Alexey S. Koshelev, Leonardo Modesto, Leslaw Rachwal and Alexei A. Starobinsky, *Occurrence of exact R2 inflation in non-local UV-complete gravity*, JHEP 1611 (2016) 067

The authors shows that the R^2 inflationary space-time is an exact solution of a range of weakly non-local (quasi-polynomial) gravitational theories, which provide an ultraviolet completion of the R^2 theory. These theories are ghost-free, super-renormalizable or finite at quantum level, and perturbatively unitary. Their spectrum consists of the graviton and the scalaron that is responsible for driving the inflation. Notably, any further extension of the spectrum leads to propagating ghost degrees of freedom.

A modification that does not assume the existence of a new dimensional parameter in the action

The following class of nonlocal gravity models has been proposed to explain current cosmic acceleration without dark energy:

$$S_2 = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G_N} \left[R \left(1 + f(\Box^{-1} R) \right) - 2\Lambda \right] + \mathcal{L}_{\rm m} \right\} , \qquad (53)$$

Here $\kappa^2 \equiv 8\pi / {M_{\rm Pl}}^2$,
the Planck mass being $M_{\rm Pl} = G^{-1/2} = 1.2 \times 10^{19}$ GeV,
g is the determinant of the metric tensor $g_{\mu\nu}$,
f is a differentiable function,
Λ is the cosmological constant,
$\mathcal{L}_{\rm m}$ is the matter Lagrangian,
$\Box$ is covariant d'Alembertian for a scalar field.
There is NO new dimensional parameter in the action: The term $\Box^{-1} R$ is dimensionless.
It can appear as a prefactor for the Newtonian gravitational constant, and explain weakening of gravity at cosmological scales.

In the FLRW metric, the d'Alembert operator acting on a scalar $A(t)$ can be expressed as

$$\Box A \equiv \frac{1}{\sqrt{-g}}\,\partial_\rho\left(\sqrt{-g}\,g^{\rho\sigma}\partial_\sigma\right)A = -\frac{1}{a^3}\frac{d}{dt}\left(a^3\frac{dA}{dt}\right)\,,$$

while its inverse operator reduces to a double integration:

$$\Box^{-1}\left[A(t)\right] = -\int\limits_{\tilde{t}_0}^{t}\frac{d\tilde{t}}{a^3(\tilde{t})}\int\limits_{\eta_0}^{\tilde{t}} d\eta a^3(\eta)A(\eta)\,.$$

where $\tilde{t}_0$ and η_0 are two initial boundaries for the integrals.
The authors determine the inverse d'Alembert operator using the retarded Green function, in other words, they fix a solution of the equation $\Box R = 0$ putting $\tilde{t}_0 = 0$ and $\eta_0 = 0$.
S. Deser, R.P. Woodard, Phys. Rev. Lett. **99** (2007) 111301, `arXiv:0706.2151`
This model is ghost-free with such choice of the Green function
S. Deser, R.P. Woodard, `arXiv:1307.6639`.

In the spatially flat FLRW metric, the independent components of field equations are:

$$\begin{aligned} 3H^2 + \Delta G_{00} &= \kappa^2 \rho_{\rm m} + \Lambda, \\ -2\dot{H} - 3H^2 + \frac{1}{3a^2}\delta^{ij}\Delta G_{ij} &= \kappa^2 P_{\rm m} - \Lambda, \end{aligned} \tag{54}$$

where

$$\begin{aligned} \Delta G_{00} &= \left[3H^2 + 3H\partial_t\right]\left\{f\left(\Box^{-1}R\right) + \Box^{-1}\left[R\frac{df}{d(\Box^{-1}R)}\right]\right\} \\ &+ \frac{1}{2}\partial_t\left(\Box^{-1}R\right)\partial_t\left(\Box^{-1}\left[R\frac{df}{d(\Box^{-1}R)}\right]\right), \\ \Delta G_{ij} &= a^2\delta_{ij}\left[\frac{1}{2}\partial_t\left(\Box^{-1}R\right)\partial_t\left(\Box^{-1}\left[R\frac{df}{d(\Box^{-1}R)}\right]\right)\right. \\ &- \left.\left[2\dot{H} + 3H^2 + 2H\partial_t + \partial_t^2\right]\left\{f\left(\Box^{-1}R\right) + \Box^{-1}\left[R\frac{df}{d(\Box^{-1}R)}\right]\right\}\right]. \end{aligned}$$

this equations get local form if we write

$$\psi(t) = \Box^{-1}R\,, \qquad \xi(t) = \Box^{-1}\left[R\frac{df}{d(\Box^{-1}R)}\right].$$

The nonlocal action (53) can be rewritten in the "localized" form by introducing two scalar fields ϕ and ξ:

$$\tilde{S}_2 = \int d^4x \frac{\sqrt{-g}}{16\pi G_N} \left\{ \left[R\left(1 + f(\eta)\right) + \xi\left(\Box\eta - R\right) - 2\Lambda \right] + \mathcal{L}_{\rm m} \right\} . \quad (55)$$

By the variation over ξ, we obtain $\Box\phi = R$.
Substituting $\phi = \Box^{-1}R$ into (55), one reobtains action (53).

S. Nojiri, S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821, arXiv:0708.0924

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S. Nojiri, S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821, arXiv:0708.0924

We should note that the initial nonlocal model maybe is not equivalent to its local formulation.
N.A. Koshelev, *Grav. Cosmol.* **15** (2009) 220, arXiv:0809.4927.
C. Deffayet and R.P. Woodard, *J. Cosmol. Astropart. Phys.* **0908** (2009) 023, arXiv:0904.0961
This non-equivalence is not a difference in the equations,
but in the initial conditions.

By varying action (55) with respect to ξ and ψ, respectively, one obtains the field equations

$$\Box\psi = R\,, \qquad \Box\xi = f'(\psi)R\,,$$

where the prime denotes derivative with respect to ψ.
The Einstein equations

$$\begin{aligned} &\frac{1}{2}g_{\mu\nu}\left[R\Psi + \partial_\rho\xi\partial^\rho\psi - 2(\Lambda + \Box\Psi)\right] - R_{\mu\nu}\Psi - \\ &- \frac{1}{2}\left(\partial_\mu\xi\partial_\nu\psi + \partial_\mu\psi\partial_\nu\xi\right) + \nabla_\mu\partial_\nu\Psi = -\kappa^2 T_{\mathrm{m}\,\mu\nu}\,, \end{aligned} \tag{56}$$

where $\Psi = 1 + f(\psi) + \xi$, the energy–momentum tensor of matter $T_{\mathrm{m}\,\mu\nu}$ is defined as

$$T_{\mathrm{m}\,\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{m}}\right)}{\delta g^{\mu\nu}}. \tag{57}$$

The function $f(\psi)$ can be determined up to a constant, because one can add it to $f(\psi)$ and subtract the same constant from ξ.

In the spatially flat FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

for a perfect matter fluid, the Einstein equations are

$$3H^2\Psi = -\frac{1}{2}\dot{\xi}\dot{\psi} - 3H\dot{\Psi} + \Lambda + \kappa^2\rho_{\rm m}, \tag{58}$$

$$\left(2\dot{H} + 3H^2\right)\Psi = \frac{1}{2}\dot{\xi}\dot{\psi} - \ddot{\Psi} - 2H\dot{\Psi} + \Lambda - \kappa^2 P_{\rm m}, \tag{59}$$

$$\ddot{\xi} = -3H\dot{\xi} - 6\left(\dot{H} + 2H^2\right)f'(\psi), \tag{60}$$

$$\ddot{\psi} = -3H\dot{\psi} - 6\left(\dot{H} + 2H^2\right), \tag{61}$$

The continuity equation is

$$\dot{\rho}_{\rm m} = -3H(P_{\rm m} + \rho_{\rm m}). \tag{62}$$

Adding up (58) and (59), we obtain the following second order linear differential equation for Ψ:

$$\ddot{\Psi} + 5H\dot{\Psi} + \left(2\dot{H} + 6H^2\right)\Psi - 2\Lambda + \kappa^2(P_{\rm m} - \rho_{\rm m}) = 0. \tag{63}$$

The choice of $f(\psi)$

For the local formulation, a reconstruction procedure has been made in
T.S. Koivisto, *Phys. Rev.* D **77** (2008) 123513, [arXiv:0803.3399]
and
E. Elizalde, E.O. Pozdeeva, S.Yu.V, *Class. Quantum Grav.* **30** (2013) 035002, arXiv:1209.5957.

This procedure shows that the simplest choice of a such function $f(\psi)$ that the model has de Sitter and power-law solutions is

$$f(\psi) = f_0 e^{\alpha\psi} ,$$

with f_0 and α nonzero real parameters.

It is also the better studied case among all possible functions $f(\psi)$:
S. Nojiri, S.D. Odintsov, *Phys. Lett.* B **659** (2008) 821,
S. Jhingan, S. Nojiri, S.D. Odintsov, M. Sami, I. Thongkool, S. Zerbini, *Phys. Lett.* B **663** (2008) 424,
T.S. Koivisto, *Phys. Rev.* D **77** (2008) 123513,
S. Nojiri, S.D. Odintsov, M. Sasaki, Y.l. Zhang, *Phys. Lett.* B **696** (2011) 278,
Y.l. Zhang, M. Sasaki, *Int. J. Mod. Phys. D* **21** (2012) 1250006,
E. Elizalde, E.O. Pozdeeva, S.Yu. V., *Phys. Rev.* D **85** (2012) 044002, ...

To reconstruct $f(\psi)$ and get a model with the exact solution for the given $H(t)$ and $w_{\rm m}(t)$ we can use the following algorithm:

- Assume the explicit form of $H(t)$ and $w_m(t)$.
- Solve **linear** equation (62) and get $\rho_{\rm m}(t)$.
- Solve **linear** equation (61) and get $\psi(t)$.
- Using $H(t)$, $w_{\rm m}(t)$, and $\rho_{\rm m}(t)$, solve **linear** equation (63) and get $\Psi(t)$.
- Substituting $\xi(t) = \Psi(t) - f(\psi) - 1$ into Eq. (60), we get a linear differential equation for $f(\psi)$:

$$\dot{\psi}^2 f''(\psi) - 12\left(\dot{H} + 2H^2\right) f'(\psi) = \ddot{\Psi} + 3H\dot{\Psi}\,. \qquad (64)$$

  To get (64) we also use *the inverse function* $t(\psi)$.
- Solve **linear** equation (64) and get the sought-for function $f(\psi)$.
- Substitute the obtained function $f(\psi)$ to Eq. (58) to check the existence of the solutions in the given form.

Note that equation (64) is a necessary condition that the model has the solutions in the given form.

Models with de Sitter solutions

Assuming that the Hubble parameter is a nonzero constant: $H = H_0$ we obtain

$$\psi(t) = -4H_0(t - t_0) - \psi_0 e^{-3H_0(t-t_0)}, \tag{65}$$

t_0, ψ_0 are integration constants. Without loss of generality we set $t_0 = 0$. Considering $w_\mathrm{m} \equiv P_\mathrm{m}/\varrho_\mathrm{m} = const \neq -1$ we obtain

$$\varrho_\mathrm{m} = \varrho_0 \, e^{-3(1+w_\mathrm{m})H_0 t}, \tag{66}$$

where ϱ_0 is an arbitrary constant.

Equation (63) has the following form and solutions:

$$\ddot{\Psi} + 5H_0\dot{\Psi} + 6H_0^2\Psi = 2\Lambda - \kappa^2(w_{\rm m} - 1)\varrho_0\, e^{-3(1+w_{\rm m})H_0 t},$$

At $w_{\rm m} \neq 0$ and $w_{\rm m} \neq -1/3$,

$$\Psi_1(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{\kappa^2\rho_0(w_{\rm m} - 1)}{3H_0^2 w_{\rm m}(1 + 3w_{\rm m})} e^{-3H_0(w_{\rm m}+1)t},$$

At $w_{\rm m} = -\frac{1}{3}$

$$\Psi_2(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} + \frac{4\kappa^2\rho_0}{3H_0} e^{-2H_0 t}\, t,$$

At $w_{\rm m} = 0$

$$\Psi_3(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t} - 1 + \frac{\Lambda}{3H_0^2} - \frac{\kappa^2\rho_0}{H_0} e^{-3H_0 t}\, t,$$

where C_1 and C_2 are arbitrary constants.

Substituting $\xi(t) = \Psi(t) - f(\psi) - 1$ into

$$\Box\xi + f'(\psi)R = 0 \tag{67}$$

we get linear differential equation to $f(\psi)$:

$$\dot{\psi}^2 f''(\psi) + \left(\ddot{\psi} + 3H_0\dot{\psi} - 12H_0^2\right) f'(\psi) = \ddot{\Psi} + 3H_0\dot{\Psi}\,. \tag{68}$$

Therefore, the model, which is described by the local action, can have de Sitter solutions only if $f(\psi)$ satisfies Eq. (68).
To demonstrate how one can get $f(\psi)$, which admits the existence of de Sitter solutions, in the explicit form, we restrict ourselves to the case $\psi_0 = 0$. In this case, Eq. (68) has the following form:

$$16H_0^2 f''(\psi) - 24H_0^2 f'(\psi) = \Phi(\psi), \tag{69}$$

where $\Phi(\psi) = \Phi(-4H_0 t) \equiv \ddot{\Psi} + 3H_0\dot{\Psi}$.

Substituting the explicit form of $\Psi(t)$, we get

$$f_1(\psi) = \frac{C_2}{4} e^{\psi/2} + C_3 e^{3\psi/2} + C_4 - \frac{\kappa^2 \rho_0}{3(1+3w_{\rm m})H_0^2} e^{3(w_{\rm m}+1)\psi/4}\,, \quad w_{\rm m} \neq -\frac{1}{3}\,,$$

$$\tilde{f}_1(\psi) = \frac{C_2}{4} e^{\psi/2} + C_3 e^{3\psi/2} + C_4 + \frac{\kappa^2 \rho_0}{4H_0^2}\left(1 - \frac{1}{3}\psi\right) e^{\psi/2}, \qquad w_{\rm m} = -\frac{1}{3}\,,$$

where C_3 and C_4 are arbitrary constants. Note that C_2 is an arbitrary constant as well. We can put $C_4 = 0$.

Equation (58) gives the condition $C_3 = 0$.

Thus, the model has de Sitter solutions if

$$f_1(\psi) = \frac{C_2}{4} e^{\psi/2} - \frac{\kappa^2 \rho_0}{3(1+3w_{\rm m})H_0^2} e^{3(w_{\rm m}+1)\psi/4}\,, \quad w_{\rm m} \neq -\frac{1}{3}.$$

$$\tilde{f}_1(\psi) = \left[\frac{\tilde{C}_2}{4} - \frac{\kappa^2 \rho_0}{12H_0^2}\psi\right] e^{\psi/2}, \qquad w_{\rm m} = -\frac{1}{3}\,,$$

Models with de Sitter solutions

E. Elizalde, E.O. Pozdeeva, and S.Yu. V.,
Phys. Rev. D **85** (2012) 044002, [arXiv:1110.5806]

Assuming that the Hubble parameter is a nonzero constant: $H = H_0$ we obtain that the model has de Sitter solutions if

$$f_1(\psi) = \frac{C_2}{4} e^{\psi/2} - \frac{\kappa^2 \rho_0}{3(1+3w_{\mathrm{m}})H_0^2} e^{3(w_{\mathrm{m}}+1)\psi/4}, \quad w_{\mathrm{m}} \neq -\frac{1}{3}.$$

$$\tilde{f}_1(\psi) = \left[\frac{\tilde{C}_2}{4} - \frac{\kappa^2 \rho_0}{12 H_0^2}\psi\right] e^{\psi/2}, \qquad w_{\mathrm{m}} = -\frac{1}{3},$$

w_{m} is a constant.
The function $f(\psi)$ can be determined up to a constant, because one can add it to $f(\psi)$ and subtract the same constant from ξ.

Power-law solutions in General Relativity

To specify different types of cosmic fluids one uses a relation between the pressure p_m and the energy density ϱ_m

$$p_\mathrm{m} = w_\mathrm{m}\varrho_\mathrm{m},$$

where w_m is the equation-of-state (EoS) parameter.
In the spatially flat FLRW metric:

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

where $a(t)$ is the scale factor, the Hubble parameter $H \equiv \dot{a}/a$,

$$w_\mathrm{m}(t) = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}.$$

For $H = \frac{C}{t-t_0}$,

$$w_\mathrm{m} = -1 + \frac{2}{3C} = CONST.$$

So, in the GR model with a barotropic perfect fluid, the Hubble parameter evolves as (let $t_0 = 0$):

$$H(t) = \frac{2}{3(w_\mathrm{m}+1)t} \equiv -\frac{2}{nt}.$$

Models with power-law solutions

E. Elizalde, E.O. Pozdeeva, and S.Yu.V.,
Class. Quantum Grav. **30** (2013) 035002, [arXiv:1209.5957]
For $H = n/t$, we get that the model with

$$f(\psi) = \Lambda \tilde{f}_1 e^{\alpha_1 \psi} + \rho_0 \tilde{f}_2 \kappa^2 e^{\alpha_2 \psi} + C_1 \tilde{f}_3 e^{\alpha_3 \psi}, \qquad (70)$$

where $\tilde{f}_i$ and α_i are constants, has solutions with $H = n/t$.
C_1 is an arbitrary constant.
The constants are subject to the following conditions:

$$\tilde{f}_1 = \frac{t_0^2}{6n(1+n)}, \qquad \alpha_1 = \frac{1-3n}{3n(2n-1)},$$

$$\tilde{f}_2 = -\frac{t_0^{2-3n-3nw_{\mathrm{m}}}}{3n(n-2+3nw_{\mathrm{m}})}, \qquad \alpha_2 = \frac{(3n(1+w_{\mathrm{m}})-2)(3n-1)}{6n(2n-1)},$$

$$\tilde{f}_3 = \frac{(n-1)t_0^{-2n}}{2(2n-1)}, \qquad \alpha_3 = \frac{3n-1}{3(2n-1)}.$$

The method allows not only to get the suitable function $f(\psi)$, but also to obtain solutions in explicit form:

$$H(t) = \frac{n}{t}, \qquad \rho_{\rm m}(t) = \rho_0 t^{-3n(w_{\rm m}+1)},$$

$$\psi(t) = -\frac{6n(2n-1)}{3n-1}\ln\left(\frac{t}{t_0}\right), \qquad \xi(t) = \Psi(t) - f(\psi) - 1,$$

$$\Psi(t) = \begin{cases} \Theta + \dfrac{\Lambda t^2}{(n+1)(3n+1)}, & n \neq -\dfrac{1}{3}, \\ \Theta + \dfrac{3}{2}\Lambda t^2\left(\ln(t) - \dfrac{3}{4}\right), & n = -\dfrac{1}{3}, \end{cases}$$

$$\Theta \equiv C_1 t^{-2n} + C_2 t^{1-3n} - \frac{\rho_0\kappa^2(w_{\rm m}-1)t^{2-3(1+w_{\rm m})n}}{(3nw_{\rm m}-1)(n+3nw_{\rm m}-2)}.$$

Power-law solutions for the function

$$f = f_0 e^{\alpha\psi}.$$

have been considered in
E. Elizalde, E.O. Pozdeeva, S.Yu.V., Y.-l. Zhang ,
JCAP **1307** (2013) 034, arXiv:1302.4330.

Power-law solutions

For power-law solutions $H = n/t$, Eq. (62) has the following general solution:

$$\rho_{\mathrm{m}}(t) = \rho_0 t^{-3n(w_{\mathrm{m}}+1)}, \tag{71}$$

where ρ_0 is an arbitrary constant.
Inserting $H = n/t$ into Eq. (61), the following solution $\psi(t)$ is obtained,

$$\psi(t) = \psi_1 t^{1-3n} - \frac{6n(2n-1)}{3n-1} \ln\left(\frac{t}{t_0}\right), \tag{72}$$

where ψ_1 and t_0 are integration constants. We consider real solutions at $t > 0$, hence, $t_0 > 0$. Note that this solution is valid provided $n \neq 1/3$ and $n \neq 1/2$.
We have shown that there is no solution for $\psi_1 \neq 0$.
So, $\psi_1 = 0$ and

$$f(\psi(t)) = f_0 \left(\frac{t}{t_0}\right)^m, \qquad m \equiv -6\alpha \frac{n(2n-1)}{3n-1}.$$

From (60) one obtains the following expression for $\xi(t)$:

$$\xi(t) = \begin{cases} \xi_0 + \xi_1 \left(\dfrac{t}{t_0}\right)^{1-3n} + \dfrac{(3n-1)f_0}{3n+m-1}\left(\dfrac{t}{t_0}\right)^{m}, & \text{for } m \neq 1-3n, \\ \xi_2 - m\, f_0 \left(\dfrac{t}{t_0}\right)^{m} \ln\left(\dfrac{t}{t_1}\right), & \text{for } m = 1-3n, \end{cases}$$

where ξ_0, ξ_1, ξ_2, and t_1 are integration constants.
Furthermore, substituting the obtained solutions into Eqs. (58) and (59), we get constraints on the integration constants.
We have found all power-law solutions in the Jordan frame.
Note that power-law solutions exist both for $\Lambda = 0$ and $\Lambda \neq 0$.
Elizalde E., Pozdeeva E.O., Vernov S.Yu., Zhang Y.-l.,
J. Cosmol. Astropart. Phys. **1307** (2013) 034, `arXiv:1302.4330`.

colorblueThank you