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Relations Between Algorithmic Reducibilities of Algebraic Systems

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Abstract—In this paper we adduce examples of various algorithmic reducibilities of algebraic systems.

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We say that a countable algebraic system $\mathfrak{A}$ is *weakly reducible* to a countable algebraic system $\mathfrak{B}$ ($\mathfrak{A} \leq_w \mathfrak{B}$) if for every isomorphic copy $\mathfrak{B}'$ of the system $\mathfrak{B}$ whose universe $|\mathfrak{B}'|$ consists of natural numbers there exists an isomorphic copy $\mathfrak{A}'$ of the system $\mathfrak{A}$ whose universe $|\mathfrak{A}'|$ also consists of natural numbers such that the atomic diagram $D(\mathfrak{A}')$ is Turing reducible to the atomic diagram $D(\mathfrak{B}')$. If the reducibility of atomic diagrams is realized by a fixed Turing operator, then we say that the system $\mathfrak{A}$ is *strongly reducible* to the system $\mathfrak{B}$ ($\mathfrak{A} \leq_s \mathfrak{B}$). Replacing the Turing reducibility of atomic diagrams with the enumeration reducibility, we get the notion of the *weak e-reducibility* ($\mathfrak{A} \leq_{we} \mathfrak{B}$) and in the case of a fixed enumeration operator we do the notion of the *strong e-reducibility* ($\mathfrak{A} \leq_{se} \mathfrak{B}$).

It is clear that the strong (e-)reducibility implies the weak (e-)reducibility. A. I. Stukachev [1] has shown that the weak (strong) e-reducibility implies the weak (strong) reducibility, while the strong e-reducibility of a system $\mathfrak{A}$ to a system $\mathfrak{B}$ takes place if the system $\mathfrak{A}$ is Σ-definable without parameters in a hereditary finite set of the system $\mathfrak{B}$. In addition, he noted that if a system has a degree then all mentioned reducibilities to this system (including the Σ-definability) are equivalent up to a finite constant enrichment.

The following theorem witnesses that in a general case there exist no other relationships between reducibilities.

Theorem. *There exist countable algebraic systems* $\mathfrak{A}_k$, $k = \overline{1,5}$, *and a countable algebraic system* $\mathfrak{B}_k$ *such that the following statements hold*:

(1) $\mathfrak{B} \leq_{se} \mathfrak{A}_1$ *and* $\mathfrak{B}$ *is not* Σ*-definable in* $\mathbb{HF}(\mathfrak{A}_1)$;

(2) $\mathfrak{A}_2 \leq_s \mathfrak{B}$ *and* $\mathfrak{A}_2 \not\leq_{we} \mathfrak{B}$;

(3) $\mathfrak{A}_3 \leq_{we} \mathfrak{B}$ *and* $\mathfrak{A}_3 \not\leq_s (\mathfrak{B}, \overrightarrow{b})$;

(4) $\mathfrak{A}_4 \leq_{we} \mathfrak{B}$, $\mathfrak{A}_4 \leq_s \mathfrak{B}$ *and* $\mathfrak{A}_4 \not\leq_{se} (\mathfrak{B}, \overrightarrow{b})$;

(5) $\mathfrak{A}_5 \leq_w \mathfrak{B}$, $\mathfrak{A}_5 \not\leq_{we} \mathfrak{B}$ *and* $\mathfrak{A}_5 \not\leq_s (\mathfrak{B}, \overrightarrow{b})$.

In (3)–(5) $\overrightarrow{b}$ *is an arbitrary finite array of elements of* $\mathfrak{B}$.

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