Zeros of the Riemann zeta function and differential operators

V. V. Kapustin

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The Riemann Hypothesis and selfadjoint operators

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The result

The Main Theorem

Let A be the Schrödinger operator $Au = -u'' + qu$ on the half-line $(x_0, +\infty)$ with

$$x_0 = \log(4\pi), \qquad q(x) = \frac{1}{4}e^{2x} \pm \frac{1}{2}e^{x}$$

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and with a selfadjoint boundary condition at x_0 so that the spectrum of A does not contain the point 0.

There exists a rank-one perturbation of A^{-1}, whose spectrum coincides with the set

$$\left\{\frac{4}{z^2}: \quad |\mathrm{Im}\, z| < \frac{1}{2}, \quad \zeta\left(\frac{1}{2} + iz\right) = 0\right\} \setminus \left\{\frac{4}{\nu^2}\right\},$$

where ν is a real number such that ζ has a simple zero at $\frac{1}{2} + i\nu$.

Remarks

- The property of the spectrum is a weaker form of the assertion that the spectrum of a rank-one perturbation of A is the set

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  "a toy model": the spetrum looks like a regularized set of zeta zeros
- For $+$, $u(x_0) = 0$ the spectrum of A is $\left\{z^2 : \ K_{\frac{1}{2}+iz}(2\pi) + K_{\frac{1}{2}-iz}(2\pi) = 0\right\}$, where K_s is the modified Bessel function

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- The Riemann zeta function and the de Branges space
- The canonical system
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$H^2 \subset L^2(\mathbb{R})$:

- H^2 is the closed linear span of $(z-\lambda)^{-1}$ with $\operatorname{Im} \lambda < 0$
- $H^2 = \mathcal{F} L^2(0, +\infty)$, where $\mathcal{F}$ is the Fourier transform $(\mathcal{F} L^2 = L^2)$

We have $L^2 = H^2 \oplus \overline{H^2}$

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Then the de Branges space $\mathcal{H}_{\mathcal{E}}$ consists of all entire functions F, for which $\frac{F}{\mathcal{E}}$ and $\frac{F^\sharp}{\mathcal{E}}$ belong to $H^2(\{\mathrm{Im}\, z > 0\})$, where $F^\sharp(z) = \overline{F(\bar{z})}$

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$\omega = \frac{\mathcal{E}^\sharp}{\mathcal{E}}$ is inner, $\;F \in \mathcal{H} \iff \frac{F}{\mathcal{E}} \in H^2 \ominus \omega H^2$

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The abstract definition:

(db1) the functional $F \mapsto F(\lambda)$ is continuous for all complex λ;

(db2) the mapping $F \mapsto F^\sharp$ isometrically acts on $\mathcal{H}$;

(db3) if $F(\lambda) = 0$, then $\frac{z-\bar{\lambda}}{z-\lambda} F \in \mathcal{H}$, its norm coincides with the norm of F

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(db4) * the mapping $F \mapsto F(-z)$ isometrically acts on $\mathcal{H}$ $\quad (\mathcal{E}^\sharp(z) = \mathcal{E}(-z))$

The last property allows us to consider the even and the odd subspace of $\mathcal{H}$

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The integral defines the Mellin transform of the bounded function $\left\{\frac{1}{y}\right\}$ on $(0, 1)$, which gives us a function from $H^2(\{\operatorname{Re} s > \frac{1}{2}\})$. $\quad (s = \frac{1}{2} - iz)$

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Proposition

The function $\frac{s-1}{s^2}\zeta(s)$ belongs to $H^2(\{\mathrm{Re}\, s > \frac{1}{2}\})$

Toeplitz kernels

The Toeplitz operator T_γ on H^2 with symbol $\gamma \in L^\infty$: $f \mapsto P_{H^2}\gamma f$

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For $f(s) = \frac{s-1}{s^2}\zeta(s) \in H^2(\{\mathrm{Re}\, s > \frac{1}{2}\})$ we obtain the symbol $\frac{\overline{f(s)}}{f(s)} = \left(\frac{s-1}{s}\right)^{-3} \cdot \frac{\overline{\zeta(s)}}{\zeta(s)}$

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For

$$\xi(s) = \frac{1}{2}\, s(s-1)\, \pi^{-s/2}\, \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

we have $\xi(1-s) = \xi(s)$.

The conjecture based on the approximation

For the Toeplitz operator with symbol $\pi^{\frac{1}{2}-s} \cdot \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}$, we use the approximation based on the relation

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z}{z} \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right)^{-1}$$

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Finally, we obtain the conjecture, which we managed to prove

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$K_s^{\sharp}(2\pi) = K_{s-1}(2\pi)$ implies $\mathcal{E}^{\sharp}(z) = \mathcal{E}(-z)$, hence $F \mapsto F(-z)$ is an isometric involution on $\mathcal{H}_{\mathcal{E}}$. We obtain the even and the odd subspace of $\mathcal{H}_{\mathcal{E}}$

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The proof of the Theorem is based on the following Lemma.

Lemma (R. Romanov)

The function $\frac{2\pi^s}{\Gamma(s)} K_s(2\pi)$ is bounded and boundedly invertible in the area $\{\mathrm{Re}\, s > \frac{1}{2}\}$ and it tends to 1 as $s \to \infty$

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For the proof of the Lemma write $\frac{2\pi^s}{\Gamma(s)} K_s(2\pi) = \pi^s\,\Gamma(1-s)\Big(I_{-s}(2\pi) - I_s(2\pi)\Big)$ and use the relation $I_s(2\pi) = \frac{\pi^s}{\Gamma(1+s)}\left(1 + O\left(\frac{1}{s}\right)\right)$ for s and $-s$

The operator on the de Branges space

Take α with $\operatorname{Re}\alpha = \frac{1}{2}$, $\zeta(\alpha) = 0$, and define

$$\varphi(z) = \frac{|\operatorname{Im}\alpha|^2}{\xi\left(\frac{1}{2}\right)} \cdot \frac{\xi\left(\frac{1}{2} - iz\right)}{\left(\frac{1}{2} - iz - \alpha\right)\left(\frac{1}{2} - iz - \bar{\alpha}\right)}$$

Then φ is even; $\varphi(0) = 1$; the set of zeros of φ coincides with the image on the real line of the set of all non-trivial zeros of ζ except α and $\bar{\alpha}$

Define the (bounded) operator on $\mathcal{H}_{\mathcal{E}}$ by

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For $\lambda \neq 0$, if $\varphi(\lambda) = 0$ then $\frac{\varphi}{z-\lambda}$ is an element of $\mathcal{H}_{\mathcal{E}}$, moreover,

$$\frac{\varphi}{z - \lambda} \to \frac{\frac{\varphi}{z-\lambda} + \frac{1}{\lambda}\varphi}{z} = \frac{1}{\lambda}\frac{\varphi}{z - \lambda}$$

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The square of the operator acts as

$$F \mapsto \frac{F - \big(F(0) + zF'(0)\big)\varphi}{z^2}$$

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acts on $\mathcal{H}_{\mathcal{E}}$, its point spectrum is $\{\lambda^{-1} : \varphi(\lambda) = 0\}$; φ is even

The square of the operator acts as

$$F \mapsto \frac{F - \big(F(0) + zF'(0)\big)\varphi}{z^2}$$

If F is even then $F \to \frac{F - F(0)\varphi}{z^2}$, and if F is odd then $F \to \frac{F - F'(0)\, z\varphi}{z^2}$

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These operators are rank-one perturbations of $F \mapsto \frac{F - F(0)\psi}{z^2}$ and $F \to \frac{F - F'(0)\, z\psi}{z^2}$, resp., and there are selfadjoint operators that can be written in this form with even ψ

Canonical systems: definition

The canonical system is the equation

$$J\dot{f}(t) = zH(t)f(t)$$

on an interval, where $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$ is a function of t, z is the spectral parameter, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, H is the Hamiltonian, which is a locally summable non-negative matrix–valued (2×2) function

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$$H(t) = \begin{pmatrix} w_+(t) & 0 \\ 0 & w_-(t) \end{pmatrix}$$

(which means that the mapping $F \mapsto F(-z)$ is an isometry on the corresponding $\mathcal{H}_\mathcal{E}$)

Let $A(t,z), B(t,z)$ be entire functions of z for every t so that $\begin{pmatrix} A \\ B \end{pmatrix}$ is a solution of the canonical system with

- $A(t,z) \to 1$ and $B(t,z) \to 0$ as $t \to -\infty$ for each z

Canonical systems: the structure function

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For $\mathcal{E}(z) = E(a,z)$, the de Branges space $\mathcal{H}_{\mathcal{E}}$ is associated with the canonical system on $(-\infty, a)$

The unitary equivalence

The Hilbert space of a canonical system is the weighted space of pairs

$$\left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \; f_\pm \in L^2\Big((-\infty, a), w_\pm\Big) \right\}$$

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For $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$ define $(Vf)(z) = \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{a} \Big(f_+(t)\, A(t,z)\, w_+(t) + f_-(t)\, B(t,z)\, w_-(t) \Big)\, dt$

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Vf is an entire function, it is an element of the de Branges space corresponding to the canonical system with $\mathcal{E}(z) = E(a,z) = A(a,z) + iB(a,z)$

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$\begin{pmatrix} f_+ \\ 0 \end{pmatrix} \to$ the even subspace of $\mathcal{H}_{\mathcal{E}}$, $\begin{pmatrix} 0 \\ f_- \end{pmatrix} \to$ the odd subspace

The chain of de Branges spaces

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$$E(t,z) = A(t,z) + iB(t,z) = \sqrt{\tfrac{-t}{2\pi}} \Big((e^{-t} + e^{t})\, K_s(-t) + (e^{-t} - e^{t})\, K_{s-1}(-t) \Big)$$

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Theorem 2

The function E is the structure function of the chain of de Branges spaces for the canonical system with diagonal Hamiltonian $\begin{pmatrix} \frac{e^{2t}}{-2t} & 0 \\ 0 & \frac{e^{-2t}}{-2t} \end{pmatrix}$ on $\{t < 0\}$

$$E(t,z) = \sqrt{\frac{-t}{2\pi}}\Big((e^{-t}+e^{t})\, K_s(-t) + (e^{-t}-e^{t})\, K_{s-1}(-t) \Big)$$

$t = -2\pi$:

$$E(-2\pi,z) = \Big((e^{2\pi}+e^{-2\pi})\, K_s(2\pi) + (e^{2\pi}-e^{-2\pi})\, K_{s-1}(2\pi) \Big)$$

$$= \frac{1}{(1-|\beta|^2)^{1/2}} \left(\mathcal{E} - \bar{\beta}\mathcal{E}^{\sharp}\right) = \mathcal{E}_{\beta}, \qquad \mathcal{E}(z) = 2K_s(2\pi), \quad |\beta| < 1$$

The de Branges spaces $\mathcal{H}_{\mathcal{E}}$ and $\mathcal{H}_{\mathcal{E}_\beta}$ coincide.

The de Branges space

$$E(t,z)=\sqrt{\frac{-t}{2\pi}}\Big((e^{-t}+e^{t})\,K_s(-t)+(e^{-t}-e^{t})\,K_{s-1}(-t)\Big)$$

$t=-2\pi$:

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Now we use the unitary transformation for constructing a unitary equivalent operator in the space of the canoncal system

The unitarily equivalent operators

The operator $\mathcal{L}$ of the canonical system:

$$(\mathcal{L}f)(t) = H(t)^{-1}J\dot{f}(t) = \begin{pmatrix} w_+^{-1} & 0 \\ 0 & w_-^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{f}_+ \\ \dot{f}_- \end{pmatrix} = \begin{pmatrix} -w_+^{-1}\dot{f}_- \\ w_-^{-1}\dot{f}_+ \end{pmatrix}$$

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Define the selfadjoint operator L_0 as the restriction of $\mathcal{L}$ to the domain with boundary condition $f_+(a) = 0$. Set $T_0 = L_0^{-1}$

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$$(VT_0 f)(z) = \frac{1}{z}\left(F(z) - F(0)\,\frac{\mathcal{E} + \mathcal{E}^\sharp}{2}\right)$$

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$$(VT_0f)(z) = \frac{1}{z}\left(F(z) - F(0)\,\frac{\mathcal{E} + \mathcal{E}^\sharp}{2}\right)$$

The operator $T_0 + (*,\ e)\,\omega$ is unitarily equivalent to the operator on $\mathcal{H}_\mathcal{E}$ defined by

$$F \mapsto \frac{F - F(0)\varphi}{z} \qquad \text{with} \quad \varphi = \frac{\mathcal{E} + \mathcal{E}^\sharp}{2} - \sqrt{\pi}\, z\, V\omega$$

The result on the space of the canonical system

Theorem 3

There exists a real-valued function γ on $(-\infty,-2\pi)$, for which $\begin{pmatrix}0\\ \gamma\end{pmatrix}$ is an element of the space of the canonical system on $(-\infty,-2\pi)$ with Hamiltonian $(*)$ and such that the spectrum of the operator $T = T_0 + \left(*,\ \begin{pmatrix}1\\ 0\end{pmatrix}\right)\begin{pmatrix}0\\ \gamma\end{pmatrix}$ has the form

$\left\{\frac{1}{z}:\quad \frac{1}{2}-iz \in Z = \{s:\ \mathrm{Re}\, s\in(0,1),\ \zeta(s)=0\}\setminus\{\alpha,\bar\alpha\}\right\}$,

where α is a simple zero of ζ, $\zeta(\alpha)=0$, with $\mathrm{Re}\,\alpha=\frac{1}{2}$

$$\varphi(z) = \frac{|\mathrm{Im}\,\alpha|^2}{\xi\left(\frac{1}{2}\right)}\cdot\frac{\xi\left(\frac{1}{2}-iz\right)}{\left(\frac{1}{2}-iz-\alpha\right)\left(\frac{1}{2}-iz-\bar\alpha\right)},\qquad \omega=\begin{pmatrix}0\\ \gamma\end{pmatrix}$$

$$\varphi = \frac{\mathcal{E}+\mathcal{E}^\sharp}{2} - \sqrt{\pi}\, z\, V\omega \iff V\omega = \frac{1}{\sqrt{\pi}\, z}\left(\frac{\mathcal{E}+\mathcal{E}^\sharp}{2}-\varphi\right)\in\mathcal{H}_{\mathcal{E}}$$

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$$\varphi(z) = \frac{|\mathrm{Im}\, \alpha|^2}{\xi\left(\frac{1}{2}\right)} \cdot \frac{\xi\left(\frac{1}{2} - iz\right)}{\left(\frac{1}{2} - iz - \alpha\right)\left(\frac{1}{2} - iz - \bar{\alpha}\right)}, \qquad \omega = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$$

$$\varphi = \frac{\mathcal{E} + \mathcal{E}^\sharp}{2} - \sqrt{\pi}\, z\, V\omega \iff V\omega = \frac{1}{\sqrt{\pi}\, z} \left(\frac{\mathcal{E} + \mathcal{E}^\sharp}{2} - \varphi \right) \in \mathcal{H}_{\mathcal{E}}$$

The Riemann Hypothesis is equivalent to the assertion that the spectrum of this operator is real.

Sturm–Liouville operators

Since $\mathcal{L}\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} -w_+^{-1}\dot{f}_- \\ w_-^{-1}\dot{f}_+ \end{pmatrix}$, we have

$$\mathcal{L}^2\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = -\begin{pmatrix} w_+^{-1}\cdot\frac{d}{dt}(w_-^{-1}\dot{f}_+) \\ w_-^{-1}\cdot\frac{d}{dt}(w_+^{-1}\dot{f}_-) \end{pmatrix} = \begin{pmatrix} \mathcal{S}_+ f_+ \\ \mathcal{S}_- f_- \end{pmatrix},$$

where $\mathcal{S}_\pm$ are the Sturm–Liouville operators

$$\mathcal{S}_\pm f = -\frac{1}{w_\pm}\cdot\frac{d\,(p_\pm f)}{dt}$$

with $w_\pm(t) = \frac{e^{\pm 2t}}{-2t}$, $\quad p_\pm(t) = w_\mp(t)^{-1} = -2t\cdot e^{\pm 2t}$; $\quad \mathcal{L}^2$ splits into a direct sum

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Boundary conditions at -2π $\rightarrow$ selfadjoint operators S_+, S_-

The Main Theorem

Let A be the Schrödinger operator $Au = -u'' + qu$ on the half-line $(x_0, +\infty)$ with

$$x_0 = \log(4\pi), \qquad q(x) = \frac{1}{4}e^{2x} \pm \frac{1}{2}e^{x},$$

and a selfadjoint boundary condition at x_0 so that the spectrum of A does not contain the point 0.

There exists a rank-one perturbation of A^{-1}, whose spectrum coincides with the set

$$\left\{ \frac{4}{z^2} : \quad |\mathrm{Im}\, z| < \frac{1}{2}, \quad \zeta\left(\frac{1}{2} + iz\right) = 0 \right\} \setminus \left\{ \frac{4}{\nu^2} \right\},$$

where ν is a real number such that ζ has a simple zero at $\frac{1}{2} + i\nu$.

The proof of the Main Theorem

Theorem 3 gives us the operator T with spectrum $\left\{ \frac{1}{z^2} : \ \frac{1}{2} + iz \in Z \right\}$

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The reduction of an arbitrary Sturm–Liouville operator to a Schrödinger operator can be realized by the Liouville transform. We use a modified version of it, which corresponds to the change of variable $t = -\frac{1}{2}e^x$. Namely, the mappings

$$f_+ \mapsto \frac{1}{\sqrt{2}}\, e^{-\frac{1}{2}e^x} f_+(-e^x/2), \qquad f_- \mapsto \frac{1}{\sqrt{2}}\, e^{\frac{1}{2}e^x} f_-(-e^x/2)$$

isometrically send the spaces $L^2\big((-\infty, -2\pi), w_\pm\big)$ onto the unweighted L^2-space on the interval $\big(\log(4\pi), +\infty\big)$, and the unitary copies of the mappings $S_\pm$ are the operators $4A$ with A as in the Theorem

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Thank you for your attention!