ON SHALEV CONJECTURE FOR SIMPLE GROUPS

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Recall that a group word $\omega = \omega(x_1, \dots, x_d)$ is an element of a free group F_d with free generators $x_1, \dots, x_d$. We may write $\omega = x_{i_1}^{m_1} \dots x_{i_k}^{m_k}$ where $i_j \in \{1, \dots, d\}$, and m_j are integers. For a group G and $g_1, \dots, g_d \in G$ we write

$$\omega(g_1, \dots, g_d) = g_{i_1}^{m_1} \dots g_{i_k}^{m_k} \in G.$$

The corresponding map $\omega : G^d \to G$ is called a word map and its image is denoted by $\omega(G)$. Many papers are devoted to estimate the size of $\omega(G)$. In [1], Larsen and Shalev proved the following

Theorem 1. *Let G be a finite simple group of Lie type and of rank n and $\omega \neq 1$ be a word. Then there exists $N = N(\omega)$ such that if G is not of type A_n or ${}^2A_n$, and $|G| \geqslant N$ then*

$$|\omega(G)| \geqslant cn^{-1}|G|$$

for some absolute constant $c > 0$.

Notice that c depends on ω in Theorem 1. Later in [2], Nikolov and Pyber found a weaker lower bound for the groups of type A_n and ${}^2A_n$. In the expository article [3], Shalev conjectured that Theorem 1 holds for all groups of Lie type.

Conjecture. [3, Conjecture 5.6]. *For every word $\omega \neq 1$ there exists a number $N = N(\omega)$ such that if G is an alternating group of degree n or a finite simple group of Lie type of rank n, and $|G| \geqslant N$, then*

$$|\omega(G)| \geqslant cn^{-1}|G|,$$

where $c > 0$ is an absolute constant.

The main result of our talk is Theorem 2.

Theorem 2. *Let $\omega \in F_d \setminus F_d'$ and $G = \mathrm{PSL}_n^{\varepsilon}(q)$. Then there exists a positive constants N, c depending only on w such that if $|G| \geqslant N$, then*

$$|\omega(G)| \geqslant c\frac{\ln(n)}{n}|G|.$$

References

[1] M. Larsen and A. Shalev. Word maps and Waring type problems. *J. Amer. Math. Soc.* **22** (2009), 437–466.

[2] N. Nikolov and L. Pyber. Product decompositions of quasirandom groups and a Jordan type theorem. *J. Eur. Math. Soc.* **13** (2011), 1063–1077.

[3] A. Shalev. Some results and problems in the theory of word maps. Erdős centennial, 611–649, Bolyai Soc. Math. Stud., 25, János Bolyai Math. Soc., Budapest, 2013.

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