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EXPLICIT SOLUTION OF SINGULAR INTEGRAL EQUATION WITH THE WEIERSTRASS ZETA FUNCTION IN THE CAPACITY OF KERNEL

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1. Let $\omega, \omega', a \in \mathbb{R}_+$, $\lambda \in \mathbb{R}$ be parameters, and $0 < a < \omega$. In the space $L^2(-a,a)$ we consider the following singular integral equation:

$$\lambda f(x) - \frac{1}{\pi i}\int_{-a}^{a} f(t)\zeta(t-x)dt = g(x), \quad -a < x < a, \tag{1}$$

where $\zeta(\cdot)$ is the Weierstrass zeta function (see [1]) with basic periods 2ω and $2i\omega'$. Since the function ζ is odd, and $\zeta(z) \sim \frac{1}{z}$ as $z \to 0$, we have that equation (1) is a complete singular integral equation with the Cauchy kernel. As is known (see [2], Chap. 1), the integral operator

$$(\mathcal{K}f)(x) := \frac{1}{\pi i}\int_{-a}^{a} f(t)\zeta(t-x)dt \tag{2}$$

acts boundedly in $L^2(-a,a)$, and the equality

$$\mathcal{K}^2 = \mathcal{I} + \mathcal{T} \tag{3}$$

is fulfilled, where $\mathcal{I}$ is the identical and $\mathcal{T}$ completely continuous operators, respectively. Operator (2) is selfadjoint, because with the given choice of periods the zeta function is real on the real axis. Our aim is *to find in explicit form its spectrum and eigenfunctions.*

With the aim to apply to equation (1) the method of analytic continuation (see [3]), we introduce the new unknown function

$$F(z) := \frac{1}{2\pi i}\int_{-a}^{a} f(t)\zeta(t-z)dt, \quad z \in \sqcap, \tag{4}$$

which is analytic in the rectangle $\sqcap := \{z = x+iy \mid -\omega \leqslant x \leqslant \omega;\ -\omega' \leqslant y \leqslant \omega'\}$ (Fig. 1), cut along the segment $[-a,a]$.

The limit values of the function F on the interval $(-a,a)$ are related via Sokhotskiĭ formulas (see [3]):

$$\begin{aligned} &F^+(x) - F^-(x) = f(x),\\ &F^+(x) + F^-(x) = \frac{1}{\pi i}\int_{-a}^{a} f(t)\zeta(t-x)dt. \end{aligned} \tag{5}$$