Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 46, No. 4, pp.11–15, 2002 *UDC 517.956*

CHARACTERISTIC PRINCIPLE OF LOCAL EXTREMUM FOR A HYPERBOLIC EQUATION AND ITS APPLICATION

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1. The Goursat problem

Let is consider the equation

$$u_{xy} + \frac{\alpha}{x+y} u_y = 0, \quad 0 < \alpha < 1, \tag{1.1}$$

on the set $G = G_- \cup G_+$, where

$$G_- = \{(x,y) \mid -h < x < 0,\ -x < y < h\}, \quad G_+ = \{(x,y) \mid 0 < x < h,\ 0 < y < h - x\}.$$

In a way similar to that in [1] (pp. 67–68), in this article, applying the Riemann method for equation (1.1) in the domains G_- and G_+, we prove the following propositions.

Theorem 1. *If*

$$u(x,h) = \omega(x) \in C_{[-h;0]} \cap C^1_{(-h;0)}, \tag{1.2}$$

$$u(0,y) = \varphi(y) \in C^1_{[0;h]}, \tag{1.3}$$

$\omega(0) = \varphi(h) = 0$, *then the Goursat problem in the domain* G_- *for equation* (1.1) *with data* (1.2), (1.3) *has the unique solution*

$$u(x,y) = \omega(x) - \int_y^h \varphi'(s) s^\alpha (x+s)^{-\alpha} ds; \tag{1.4}$$

in addition, $u(x,y) \in C(\overline{G}_-)$.

Theorem 2. *If*

$$u(x,0) = \psi(x) \in C_{[0;h]} \cap C^1_{(0;h)}, \tag{1.5}$$

$\psi(0) = \varphi(0) = 0$, *then the Goursat problem in the domain* G_+ *for equation* (1.1) *with data* (1.3), (1.5) *has the unique solution*

$$u(x,y) = \psi(x) - \int_0^y \varphi'(s) s^\alpha (x+s)^{-\alpha} ds; \tag{1.6}$$

in addition, $u(x,y) \in C(\overline{G}_+)$.

2. Characteristic principle of local extremum

Let

$$v_-(x,y) = \frac{\partial}{\partial x}\int_{-y}^{x}(x-t)^{-\lambda_1}(-t)^{r_1}u(t,y)dt,$$

where $0 < \lambda_1 < 1$, $\lambda_1 < r_1$, the function $u(t,y)$ be defined by formula (1.4).

Lemma 1. *If the function $u(x,y) \in C(\overline{G}_-)$ is such that $u(0,y) = \varphi(y)$ attains its most positive (least negative) value at the point ξ, $0 < \xi < h$, and $u(x,h) \equiv 0$, where $u(x,h)$ is defined by formula (1.2), then* $\lim\limits_{x\to 0-0} v_-(x,\xi) < 0$ (> 0).

Proof. Using formula (1.4), we find

$$v_-(x,y) = \frac{\partial}{\partial x}\Big[\int_{-y}^{x}\omega(t)(x-t)^{-\lambda_1}(-t)^{r_1}dt - \int_{-y}^{x}(x-t)^{-\lambda_1}(-t)^{r_1}dt\int_{y}^{h}\varphi'(s)s^{\alpha}(t+s)^{-\alpha}ds\Big]. \tag{2.1}$$

We integrate by parts the first addend of the expression standing in the brackets, assuming $U = \omega(t)$,

$$V = V(x,t) = \int_{t}^{x}(x-s)^{-\lambda_1}s^{r_1}ds. \tag{2.2}$$

In the second expression in the brackets of equality (2.1) we change the order of integration. We obtain

$$\int_{-y}^{x}(x-t)^{-\lambda_1}(-t)^{r_1}dt\int_{y}^{h}\varphi'(s)s^{\alpha}(t+s)^{-\alpha}ds = \int_{y}^{h}\varphi'(s)s^{\alpha}ds\int_{-y}^{x}(x-t)^{-\lambda_1}(-t)^{r_1}(t+s)^{-\alpha}dt. \tag{2.3}$$

Integral (2.2) and the interior integral in the right-hand side of formula (2.3) is calculated by means of the changes of variables $s = t + (x-t)z$ and $t = -y + (x+y)z$, respectively. The results of calculation give us

$$V(x,t) = -\frac{1}{1-\lambda_1}(x-t)^{1-\lambda_1}(-t)^{r_1}F\Big(1, -r_1; 2-\lambda_1; \frac{x-t}{-t}\Big), \tag{2.4}$$

$$\int_{-y}^{x}(x-t)^{-\lambda_1}(-t)^{r_1}(t+s)^{-\alpha}dt =$$
$$= \frac{1}{1-\lambda_1}y^{r_1}(x+y)^{1-\lambda_1}(s-y)^{-\alpha}F_1\Big(1, -r_1, \alpha; 2-\lambda_1; \frac{x+y}{y}, \frac{x+y}{y-s}\Big). \tag{2.5}$$

From equalities (2.2)–(2.5) we determine

$$v_-(x,y) = \frac{\partial}{\partial x}\Big[\frac{1}{1-\lambda_1}\omega(-y)(x+y)^{1-\lambda_1}y^{r_1}F\Big(1, -r_1; 2-\lambda_1; \frac{x+y}{y}\Big) +$$
$$+ \frac{1}{1-\lambda_1}\int_{-y}^{x}\omega'(t)(-t)^{r_1}(x-t)^{1-\lambda_1}F\Big(1, -r_1; 2-\lambda_1; \frac{x-t}{-t}\Big)dt -$$
$$- \frac{1}{1-\lambda_1}(x+y)^{1-\lambda_1}y^{r_1}\int_{y}^{h}\varphi'(s)s^{\alpha}(s-y)^{-\alpha}F_1\Big(1, -r_1, \alpha; 2-\lambda_1; \frac{x+y}{y}, \frac{x+y}{y-s}\Big)ds\Big].$$

Applying formulas of "quick" differentiation

$$\frac{d}{dz}z^{c-1}F(a,b;c;pz) = (c-1)z^{c-2}F(a,b;c-1;pz),$$
$$\frac{d}{dz}z^{c-1}F_1(a,b,b';c;pz,qz) = (c-1)z^{c-2}F_1(a,b,b';c-1;pz,qz),$$

where p and q do not depend on z, proved by V.F. Volkodavov, and the identities (see [2], pp. 73, 232)

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$
$$F_1(a,b,b';c;1;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b';c-b;z),\quad c-a-b>0,$$

we find

$$\lim_{x\to 0-0}v_-(x,y)=\frac{\lambda_1}{r_1-\lambda_1}\Big[y^{r_1-\lambda_1}\int_y^h\varphi'(s)F\Big(r_1-\lambda_1,\alpha;r_1-\lambda_1+1;\frac{y}{s}\Big)ds-$$
$$-\omega(-y)y^{r_1-\lambda_1}-\int_{-y}^0\omega'(t)(-t)^{r_1-\lambda_1}dt\Big].\quad (2.6)$$

By the condition of Lemma

$$\lim_{x\to 0-0}v_-(x,y)=\frac{\lambda_1}{r_1-\lambda_1}y^{r_1-\lambda_1}\int_y^h\varphi'(s)F\Big(r_1-\lambda_1,\alpha;r_1-\lambda_1+1;\frac{y}{s}\Big)ds.\quad (2.7)$$

Let $\varphi(y)$ attain the most positive (the least negative value) value at a point ξ, $0<\xi<h$. Using the identity $(\varphi(t)-\varphi(\xi))'=\varphi'(t)$, in equality (2.7) we integrate by parts, assuming $U=F(r_1-\lambda_1,\alpha;r_1-\lambda_1+1;\frac{y}{s})$. We obtain

$$\lim_{x\to 0-0}v_-(x,\xi)=-\frac{\lambda_1}{r_1-\lambda_1}\xi^{r_1-\lambda_1}\varphi(\xi)F\Big(r_1-\lambda_1,\alpha;r_1-\lambda_1+1;\frac{\xi}{h}\Big)+$$
$$+\frac{\alpha\lambda_1}{r_1-\lambda_1+1}\xi^{r_1-\lambda_1+1}\int_\xi^h(\varphi(s)-\varphi(\xi))s^{-2}F\Big(r_1-\lambda_1+1,\alpha+1;r_1-\lambda_1+2;\frac{\xi}{s}\Big)ds.$$

This equality implies the assertions of Lemma.

Let

$$v_+(x,y)=\frac{\partial}{\partial x}\int_x^{h-y}(t-x)^{-\lambda_2}t^{r_2}u(t,y)dt,$$

where $0<\lambda_2<1$, $\lambda_2<r_2$, the function $u(t,y)$ be defined by formula (1.6). Following the way in the proof of Lemma 1 to obtain expression (2.6), we find

$$\lim_{x\to 0+0}v_+(x,y)=\frac{\lambda_2}{r_2-\lambda_2}\Big[(h-y)^{r_2-\lambda_2}\int_0^y\varphi'(s)s^\alpha(s+h-y)^{-\alpha}F\Big(1,\alpha;r_2-\lambda_2+1;\frac{h-y}{s+h-y}\Big)ds+$$
$$+\psi(h-y)(h-y)^{r_2-\lambda_2}-\int_0^{h-y}\psi'(t)t^{r_2-\lambda_2}dt\Big].\quad (2.8)$$

With the use of equality (2.8) as in Lemma 1 we can prove

Lemma 2. *If $u(x,y)\in C(\overline{G}_+)$ is such that $u(0,y)=\varphi(y)$ attains the most positive (the least negative) value at a point ξ, $0<\xi<h$, and $u(x,0)\equiv 0$, where $u(x,0)$ is defined by formula (1.5), $r_2-\lambda_2>\alpha$, then $\lim\limits_{x\to 0+0}v_+(x,\xi)>0$ (<0).*

3. The uniqueness and existence of a solution of the problem Δ_2 for equation (1.1)

Problem Δ_2. On the set G find a solution $u(x,y) \in C(\overline{G}_+ \cap \overline{G}_-)$ of equation (1.1), which satisfies conditions (1.3), (1.4) and the condition of conjugation

$$\lim_{x\to 0-0} v_-(x,y) = b(y) \lim_{x\to 0+0} v_+(x,y), \tag{3.1}$$

where $b(y)$ is a given function, $\lim_{x\to 0-0} v_-(x,y)$ and $\lim_{x\to 0+0} v_+(x,y)$ are defined by formulas (2.6) and (2.8), respectively.

Theorem 3. *If $b(y) \in C_{(0;h)}$, $b(y) > 0$, $r_2 - \lambda_2 > \alpha$ and a solution of problem Δ_2 exists, then it is unique.*

The proof of this theorem is carried out by the method of proof by contradiction with the use of Lemmas 1 and 2.

Theorem 4. *If the function $\omega(x)$ is subordinated to the condition of Theorem 1, $\psi(x) \in C^1_{[0;h]}$, $\psi(h) = 0$, $r_i - \lambda_i - \alpha > 1$, $i = 1, 2$, $b(y) \equiv b$ ($b > 0$, b is constant), then the unique solution of problem Δ_2 exists.*

Proof. Taking into account formulas (2.6), (2.8) and the condition of conjugacy (3.1), we arrive at the equation with respect to the unknown function $\varphi'(y)$

$$\frac{\lambda_1}{r_1-\lambda_1} y^{r_1-\lambda_1} \int_y^h \varphi'(s) F\Big(r_1-\lambda_1, \alpha; r_1-\lambda_1+1; \frac{y}{s}\Big) ds -$$
$$-\frac{\lambda_2 b}{r_2-\lambda_2}(h-y)^{r_2-\lambda_2} \int_0^y \varphi'(s) s^\alpha (s+h-y)^{-\alpha} F\Big(1, \alpha; r_2-\lambda_2+1; \frac{h-y}{s+h-y}\Big) ds = g(y), \tag{3.2}$$
$$g(y) = \frac{\lambda_1}{r_1-\lambda_2}\omega(-y) y^{r_1-\lambda_1} + \frac{\lambda_1}{r_1-\lambda_1}\int_{-y}^0 \omega'(t)(-t)^{r_1-\lambda_1} dt +$$
$$+\frac{b\lambda_2}{r_2-\lambda_2}\psi(h-y)(h-y)^{r_2-\lambda_2} - \frac{b\lambda_2}{r_2-\lambda_2}\int_0^{h-y} \psi'(t) t^{r_2-\lambda_2} dt.$$

Let $\varphi'(t)$ be a solution of equation (3.2). Under the fulfillment of the conditions of Theorem we differentiate both the sides of this identity with respect to y, applying here the identity (see [2], p. 111)

$$\frac{d}{dz} z^{c-1}(1-z)^{b-c+1} F(a,b;c;z) = (c-1) z^{c-2}(1-z)^{b-c} F(a-1, b; c-1; z).$$

The differentiation of identity (3.2) leads us to the equation

$$\varphi'(y) = \int_0^h \varphi'(s) K(y,s) ds + f(y), \tag{3.3}$$
$$K(y,s) = \begin{cases} K_1(y,s), & 0 \le s \le y; \\ K_2(y,s), & y \le s \le h, \end{cases}$$
$$K_1(y,s) = -\frac{b\lambda_2}{a(y)}(h-y)^{r_2-\lambda_2-1} y^{-\alpha} s^\alpha (s+h-y)^{-\alpha}, \quad K_2(y,s) = -\frac{\lambda_1}{a(y)} y^{r_1-\lambda_1-\alpha-1}(s-y)^\alpha s^{-\alpha},$$
$$a(y) = -\lambda_1 B(r_1-\lambda_1; 1-\alpha) y^{r_1-\lambda_1-\alpha} - \frac{b\lambda_2}{r_2-\lambda_2} h^{-\alpha}(h-y)^{r_2-\lambda_2} F\Big(1, \alpha; r_2-\lambda_2+1; \frac{h-y}{h}\Big),$$
$$f(y) = -\frac{1}{a(y)} y^{-\alpha}[\lambda_1 \omega(-y) y^{r_1-\lambda_1-1} - b\lambda_2 \psi(h-y)(h-y)^{r_2-\lambda_1-1}].$$

Under the fulfillment of the conditions of this theorem $a(y), f(y) \in C_{[0;h]}$, and $a(y) < 0$, $y \in [0;h]$. One can easily see that $K(y,s)$ is a bounded function in $[0,h;0,h]$ and possesses one line of a finite discontinuity, i. e., by the definition it is a regular kernel (see [3], p. 19). In [3] (pp. 39–55) the existence and uniqueness of a solution of equation (3.3) were proved. This solution can be written via the resolvent

$$\varphi'(y) = f(y) + \int_0^h R(y, s; 1) f(s) ds,$$

which is not cited here in view of of its cumbersome form (see [3], p. 54).

The existence and uniqueness of the solution of equation (3.3) imply the existence of the solution of Problem Δ_2. Theorem 4 has been proved.

References

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28 February 2001

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