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Ranges of η-Functions of η-Like Linear Orderings

A. N. Frolov[1*]
(Submitted by M.M. Arslanov)

[1] *Kazan (Volga Region) Federal University, ul. Kremlyovskaya 18, Kazan, 420008 Russia*
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Abstract—We completely describe ranges of η-functions of η-like linear orderings without computable presentations.

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1. INTRODUCTION

A linear ordering is called *η-like*, if it can be represented as $\sum_{q\in\mathbb{Q}} f(q)$ for some function $f : \mathbb{Q} \to \mathbb{N}$ such that $0 \notin \mathrm{range}(f)$. The function f is called an *η-function* of the linear ordering L.

Computable η-like linear orderings play an important role in the theory of computable structures. In this research area there are many papers (e.g., [1–6]).

J. R. Rosenstein [1] has proved that for a computable η-like linear ordering, an η-function can be chosen from the class Δ^0_3 and hence its range belongs to the class Σ^0_3. S. Fellner [2] has shown that if $f \in \Pi^0_2$ then the linear ordering $\sum_{q\in\mathbb{Q}} f(q)$ has a computable presentation. However, not every Δ^0_3-computable function is an η-function of a computable linear ordering. The first example of such a function (with an infinite range) was constructed in the joint work by M. Lerman and J. R. Rosenstein [3]. Later R. G. Downey ([4], theorem 4.16) has constructed a Δ^0_3-computable function f such that $\mathrm{range}(f) = \{1, 2, 3, 4\}$ and the linear ordering $\sum_{q\in\mathbb{Q}} f(q)$ has no computable presentation.

It is natural to ask: What other ranges may have similar examples of functions, i.e., functions different from η-functions for computable linear orderings? In particular, is it possible to prove the result obtained by R. Downey for $\mathrm{range}(f) = \{1, 2\}$ instead of $\mathrm{range}(f) = \{1, 2, 3, 4\}$? In this paper we give positive answer to the latter question. Moreover, for any finite nonsingleton set D without zero there exists a Δ^0_3-computable function which is not an η-function of any computable linear ordering and has the range D. At the end part of this paper we consider this question for η-functions with an infinite range.

2. RANGES OF η-FUNCTIONS

Earlier we have proved the following theorem.

Theorem 1 ([6]). *Let L be an arbitrary $\mathbf{0}'$-computable linear ordering and let τ be a computable linear ordering without endpoints. Then the linear ordering $\tau \cdot L$ has a computable copy.*

Corollary. Let $a_0 < a_1 < \cdots < a_n$, $a_0 \neq 0$, and $n \geq 1$. Then a linear ordering L has a $\mathbf{0}'$-computable presentation if and only if the linear ordering $L' = (a_0\eta + a_1 + a_0\eta + \cdots + a_0\eta + a_n + a_0\eta) \cdot L$ has a computable presentation.

*E-mail: Andrey.Frolov@mail.ru.