

UDK 530.12

DIFFERENTIAL GEOMETRY OF WALKER MANIFOLDS

*A.A. Salimov, M. Iscan, S. Turanli***Abstract**

In the present paper, we focus our attention on the integrability and holomorphic conditions of a Norden–Walker structure (M, g^{N+}, φ) . We also give a characterization of a Kähler–Norden–Walker metric g^{N+} .

Key words: Norden–Walker structure, Walker manifolds, pure tensor field, Kähler–Norden–Walker metrics, holomorphic tensor field, twin metrics.

Introduction

Let M be a C^∞ -manifold of finite dimension 4. We denote by $\mathfrak{S}_s^r(M)$ the module over $F(M)$ of all C^∞ -tensor fields of type (r, s) on M , i.e., of contravariant degree r and covariant degree s , where $F(M)$ is the algebra of C^∞ -functions on M .

A neutral metric wg on a 4-manifold M is said to be Walker metric if there exists a 2-dimensional null distribution D on M , which is parallel with respect to wg . From Walker's theorem [1], there is a system of coordinates with respect to which wg takes the local canonical form

$${}^wg = ({}^wg_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \quad (1)$$

where a, b, c are smooth functions of the coordinates (x, y, z, t) . The parallel null 2-plane D is spanned locally by $\{\partial_x, \partial_y\}$, where $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$.

In [2, Fact 1], a proper almost complex structure with respect to wg is defined as a wg -orthogonal almost complex structure φ so that φ is a standard generator of a positive $\pi/2$ rotation on D , i.e., $\varphi\partial_x = \partial_y$ and $\varphi\partial_y = -\partial_x$. Then for the Walker metric wg , such a proper almost complex structure φ is determined uniquely as

$$\begin{pmatrix} 0 & -1 & -c & \frac{1}{2}(a-b) \\ 1 & 0 & \frac{1}{2}(a-b) & c \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (2)$$

In [3], for such a proper almost complex structure φ on Walker 4-manifold M , an almost Norden structure (g^{N+}, φ) is constructed, where g^{N+} is a metric on M , with properties $g^{N+}(\varphi X, \varphi Y) = -g^{N+}(X, Y)$. In fact, as one of these examples, such a metric takes the form (see Proposition 6 in [3]):

$$g^{N+} = \begin{pmatrix} 0 & -2 & 0 & -b \\ -2 & 0 & -a & -2c \\ 0 & -a & 0 & \frac{1}{2}(1-ab) \\ -b & -2c & \frac{1}{2}(1-ab) & -2bc \end{pmatrix}. \quad (3)$$

We may call this an almost Norden–Walker metric. The construction of such a structure in [3] is to find a Norden metric for a given almost complex structure, which is different from the Walker metric.

In [3], for a given proper almost complex structure φ , another Norden–Walker metric G^{N+} is also constructed:

$$G^{N+} = \begin{pmatrix} -2 & 0 & -a & -2c \\ 0 & 2 & 0 & b \\ -a & 0 & \frac{1}{2}(1-a^2) & -ac \\ -2c & b & -ac & \frac{1}{2}(b^2-4c^2-1) \end{pmatrix}. \quad (4)$$

The purpose of the present paper is to study Kähler and quasi-Kähler conditions of Norden–Walker metrics g^{N+} and G^{N+} .

1. Kähler–Norden–Walker metrics

Let φ be an affinor field on M , i.e., $\varphi \in \mathfrak{S}_1^1(M)$. A tensor field t of type (r, s) is called pure tensor field with respect to φ if

$$\begin{aligned} t(\varphi X_1, \dots, X_s; \overset{1}{\xi}, \dots, \overset{r}{\xi}) &= t(X_1, \dots, \varphi X_s; \overset{1}{\xi}, \dots, \overset{r}{\xi}) \\ &= t(X_1, \dots, X_s; \overset{1}{\varphi\xi}, \dots, \overset{r}{\varphi\xi}) \\ &\vdots \\ &= t(X_1, \dots, X_s; \overset{1}{\xi}, \dots, \overset{r}{\varphi\xi}) \end{aligned}$$

for any $X_1, X_2, \dots, X_s \in \mathfrak{S}_0^1(M)$ and $\overset{1}{\xi}, \overset{2}{\xi}, \dots, \overset{r}{\xi} \in \mathfrak{S}_1^0(M)$, where $\overset{r}{\varphi}$ is the adjoint operator of φ defined by

$$(\overset{r}{\varphi\xi})(X) = \xi(\varphi X),$$

$X \in \mathfrak{S}_0^1(M), \xi \in \mathfrak{S}_1^0(M)$.

We denote by $\mathfrak{S}_s^r(M)$ the module of all pure tensor fields of type (r, s) on M with respect to the affinor field φ . We now fix a positive integer λ . If K and L are pure tensor fields of types (p_1, q_1) and (p_2, q_2) respectively, then the tensor product of K and L with contraction

$$K \overset{C}{\otimes} L = (K_{j_1 \dots j_{q_1}}^{i_1 \dots m_\lambda \dots i_{p_1}} L_{s_1 \dots m_\lambda \dots s_{q_2}}^{r_1 \dots r_{p_2}})$$

is also a pure tensor field.

We shall now make the direct sum $\overset{*}{\mathfrak{S}}(M) = \sum_{r,s=0}^{\infty} \overset{*}{\mathfrak{S}}_s^r(M)$ into an algebra over the real number \mathbb{R} by defining the *pure product* (denoted by $\overset{C}{\otimes}$ or " \circ ") of $K \in \overset{*}{\mathfrak{S}}_{q_1}^{p_1}(M)$ and $L \in \overset{*}{\mathfrak{S}}_{q_2}^{p_2}(M)$ as follows:

$$\overset{C}{\otimes} : (K, L) \rightarrow (K \overset{C}{\otimes} L) = \begin{cases} K_{j_1 \dots j_{q_1}}^{i_1 \dots m_\lambda \dots i_{p_1}} L_{s_1 \dots m_\lambda \dots s_{q_2}}^{r_1 \dots r_{p_2}} & \text{for } \lambda \leq p_1, q_2 \\ & (\lambda \text{ is a fixed positive integer}), \\ K_{j_1 \dots m_\mu \dots j_{q_1}}^{i_1 \dots i_{p_1}} L_{s_1 \dots s_{q_2}}^{r_1 \dots m_\mu \dots r_{p_2}} & \text{for } \mu \leq p_2, q_1 \\ & (\mu \text{ is a fixed positive integer}), \\ 0 & \text{for } p_1 = 0, p_2 = 0, \\ 0 & \text{for } q_1 = 0, q_2 = 0. \end{cases}$$

In particular, let $K = X \in \mathfrak{S}_0^1(M)$, and $L \in \Lambda_q(M)$ be a q -form. Then the pure product $X \overset{C}{\otimes} L$ coincides with the *interior product* $\iota_X L$.

Definition 1 [4]. Let $\varphi \in \mathfrak{S}_1^1(M)$, and $\mathfrak{S}(M) = \sum_{r,s=0}^{\infty} \mathfrak{S}_s^r(M)$ be a tensor algebra

over \mathbb{R} . A map $\phi_\varphi : \mathfrak{S}^*(M) \rightarrow \mathfrak{S}(M)$ is called a ϕ_φ -operator on M if

- a) ϕ_φ is linear with respect to constant coefficients;
- b) $\phi_\varphi : \mathfrak{S}_s^r(M) \rightarrow \mathfrak{S}_{s+1}^r(M)$ for all r, s ;
- c) $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \overset{C}{\otimes} L + K \overset{C}{\otimes} \phi_\varphi L$ for all $K, L \in \mathfrak{S}^*(M)$;
- d) $\phi_\varphi X Y = -(L_Y \varphi) X$, $X, Y \in \mathfrak{S}_0^1(M)$, where L_Y is the Lie derivation with respect to Y ;
- e) $\phi_\varphi X (\iota_Y \omega) = (d(\iota_Y \omega))(\varphi X) - (d(\iota_Y (\omega \circ \varphi)))(X) = (\varphi X)(\iota_Y \omega) - X(\iota_Y \omega)$ for all $\omega \in \mathfrak{S}_1^0(M)$ and $X, Y \in \mathfrak{S}_0^1(M)$, where $\iota_Y \omega = \omega(Y) = \omega \overset{C}{\otimes} Y$.

Let $(M, {}^w g)$ be a Walker 4-manifold with a Norden–Walker metric g^{N+} and proper almost complex structure φ . If the Nijenhuis tensor field $N_\varphi \in \mathfrak{S}_2^1(M)$ vanishes, then φ is a complex structure and moreover M is a \mathbb{C} -holomorphic manifold $X_2(\mathbb{C})$ whose transition functions are \mathbb{C} -holomorphic mappings. $N_\varphi = 0$ is equivalent to the condition $\nabla \varphi = 0$, where ∇ is a torsion-free affine connection. A metric g^{N+} is a Norden–Walker metric [3, 5–8] if

$$g^{N+}(\varphi X, Y) = g^{N+}(X, \varphi Y) \quad (5)$$

for any $X, Y \in \mathfrak{S}_0^1(M)$, i.e., g^{N+} is pure with respect to the proper almost complex structure φ . If (M, φ) is an almost complex manifold with Norden–Walker metric g^{N+} , we say that (M, φ, g^{N+}) is an almost Norden–Walker manifold. If φ is integrable, we say that (M, φ, g^{N+}) is a Norden–Walker manifold.

Let $t \in \mathfrak{S}_s^r(X_2(\mathbb{C}))$ be a complex tensor field on $X_2(\mathbb{C})$. The real model of such a tensor field is a pure tensor field $t \in \mathfrak{S}_s^r(M)$ with respect to φ , which in general is not \mathbb{C} -holomorphic. When φ is a proper complex structure on M and the tensor field $\phi_\varphi t$ vanishes, the complex tensor field t on $X_2(\mathbb{C})$ is said to be holomorphic [9]. Thus a holomorphic tensor field t on $X_2(\mathbb{C})$ is realized on M in the form of a pure tensor field t , such that

$$(\phi_\varphi t)(X, Y_1, Y_2, \dots, Y_s, \overset{1}{\xi}, \overset{2}{\xi}, \dots, \overset{r}{\xi}) = 0$$

for any $X, Y_1, \dots, Y_s \in \mathfrak{S}_0^1(M)$ and $\overset{1}{\xi}, \overset{2}{\xi}, \dots, \overset{r}{\xi} \in \mathfrak{S}_1^0(M)$, where

$$\begin{aligned} (\phi_\varphi t)(X, Y_1, \dots, Y_s, \xi^1, \dots, \xi^r) &= (\varphi X) t(Y_1, \dots, Y_s, \xi^1, \dots, \xi^r) - \\ &- X t(\varphi Y_1, \dots, \varphi Y_s, \xi^1, \dots, \xi^r) + \sum_{\lambda=1}^s t(Y_1, \dots, (L_{Y_\lambda} \varphi) X, \dots, Y_s, \xi^1, \dots, \xi^r) - \\ &- \sum_{\mu=1}^r t(Y_1, \dots, Y_s, \xi^1, \dots, L_{\varphi X} \xi^\mu - L_X(\xi^\mu \circ \varphi), \dots, \xi^r). \end{aligned} \quad (6)$$

In a Norden–Walker (almost Norden–Walker) manifold a Norden–Walker metric g^{N+} is called *holomorphic (almost holomorphic)* if

$$\begin{aligned} (\phi_\varphi g^{N+})(X, Y, Z) &= (\varphi X)(g^{N+}(Y, Z)) - X(g^{N+}(\varphi Y, Z)) + \\ &+ g^{N+}((L_Y \varphi) X, Z) + g^{N+}(Y, (L_Z \varphi) X) = 0 \end{aligned}$$

for any $X, Y, Z \in \mathfrak{S}_0^1(M)$. If (M, φ, g^{N+}) is a Norden–Walker manifold with a holomorphic Norden–Walker metric g^{N+} , we say that (M, φ, g^{N+}) is a *holomorphic Norden–Walker manifold*.

In some aspects, holomorphic Norden–Walker manifolds are similar to Kähler–Norden–Walker manifolds. The following theorem is an analogue to the next known result: an almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi–Civita connection.

Theorem 1. *An almost Norden–Walker manifold is a holomorphic Norden–Walker manifold if and only if the proper almost complex structure φ is parallel with respect to the Levi–Civita connection of g^{N+} .*

Proof. By virtue of (5) and with $\nabla g = 0$ we have

$$g^{N+}(Z, (\nabla_Y \varphi)X) = g^{N+}((\nabla_Y \varphi)Z, X). \quad (7)$$

Using (7), we can transform (6) as follows:

$$\begin{aligned} (\Phi_\varphi g^{N+})(X, Z_1, Z_2) = & -g^{N+}((\nabla_X \varphi)Z_1, Z_2) + \\ & + g^{N+}((\nabla_{Z_1} \varphi)X, Z_2) + g^{N+}(Z_1, (\nabla_{Z_2} \varphi)X). \end{aligned} \quad (8)$$

From this we have

$$\begin{aligned} (\Phi_\varphi g^{N+})(Z_2, Z_1, X) = & -g^{N+}((\nabla_{Z_2} \varphi)Z_1, X) + \\ & + g^{N+}((\nabla_{Z_1} \varphi)Z_2, X) + g^{N+}(Z_1, (\nabla_X \varphi)Z_2). \end{aligned} \quad (9)$$

If we add (8) and (9), we find

$$(\Phi_\varphi g^{N+})(X, Z_1, Z_2) + (\Phi_\varphi g^{N+})(Z_2, Z_1, X) = 2g^{N+}(X, (\nabla_{Z_1} \varphi)Z_2). \quad (10)$$

By substituting $\Phi_\varphi g^{N+} = 0$ in (10), we find $\nabla \varphi = 0$. Conversely, if $\nabla \varphi = 0$, then the condition $\Phi_\varphi g^{N+} = 0$ follows from (8). Thus the proof is complete. \square

Remark. Recall that a Kähler–Norden–Walker manifold can be defined as a triple (M, φ, g^{N+}) which consists of a manifold M endowed with a proper almost complex structure φ and a pseudo-Riemannian Norden–Walker metric g^{N+} such that $\nabla \varphi = 0$, where ∇ is the Levi–Civita connection of g^{N+} . Therefore, there exist a one-to-one correspondence between Kähler–Norden–Walker manifolds and complex manifolds with a *holomorphic Norden–Walker metric* as they were defined in [9].

Let (M, φ, g^{N+}) be an almost Norden–Walker manifold. If

$$(\Phi_\varphi g^{N+})_{kij} = \varphi_k^m \partial_m g_{ij}^{N+} - \varphi_i^m \partial_k g_{mj}^{N+} + g_{mj}^{N+} (\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im}^{N+} \partial_j \varphi_k^m = 0, \quad (11)$$

then by virtue of Theorem 1 the triple (M, φ, g^{N+}) is called a holomorphic Norden–Walker or a Kähler–Norden–Walker manifold.

By substituting (2) and (3) in (11), we obtain

$$\begin{aligned} (\Phi_\varphi g^{N+})_{xxz} &= (\Phi_\varphi g^{N+})_{xzx} = a_x, & (\Phi_\varphi g^{N+})_{xxt} &= (\Phi_\varphi g^{N+})_{xtx} = -b_y + 2c_x, \\ (\Phi_\varphi g^{N+})_{xyy} &= (\Phi_\varphi g^{N+})_{xyy} = -a_y, & (\Phi_\varphi g^{N+})_{xyt} &= (\Phi_\varphi g^{N+})_{xty} = -b_x - 2c_y, \\ (\Phi_\varphi g^{N+})_{xzz} &= aa_x, & (\Phi_\varphi g^{N+})_{xzt} &= (\Phi_\varphi g^{N+})_{xtz} = -\frac{1}{2}(ab)_y + (ac)_x, \end{aligned}$$

$$\begin{aligned}
(\Phi_\varphi g^{N+})_{xtt} &= 4cc_x - bb_x - 2(bc)_y, & (\Phi_\varphi g^{N+})_{yxz} &= (\Phi_\varphi g^{N+})_{yzx} = a_y, \\
(\Phi_\varphi g^{N+})_{yxt} &= (\Phi_\varphi g^{N+})_{ytx} = b_x + 2c_y, & (\Phi_\varphi g^{N+})_{yyt} &= (\Phi_\varphi g^{N+})_{yty} = -b_y + 2c_x, \\
(\Phi_\varphi g^{N+})_{yzz} &= aa_y, & (\Phi_\varphi g^{N+})_{yzt} &= (\Phi_\varphi g^{N+})_{ytz} = \frac{1}{2}(ab)_x + (ac)_y, \\
(\Phi_\varphi g^{N+})_{ytt} &= 4cc_y - bb_y + 2(bc)_x, & (\Phi_\varphi g^{N+})_{zzz} &= (\Phi_\varphi g^{N+})_{zzx} = b_z - \frac{1}{2}a(a-b)_x, \\
(\Phi_\varphi g^{N+})_{zxx} &= 2(b_x - a_x), & (\Phi_\varphi g^{N+})_{zxy} &= (\Phi_\varphi g^{N+})_{zyx} = 2c_x - a_y + b_y, \\
(\Phi_\varphi g^{N+})_{zxt} &= (\Phi_\varphi g^{N+})_{ztx} = 2cb_x - \frac{1}{2}(a-b)b_y + 2c_z - ca_x - a_t + bc_x, \\
(\Phi_\varphi g^{N+})_{zyy} &= 4c_y, & (\Phi_\varphi g^{N+})_{zyz} &= (\Phi_\varphi g^{N+})_{zzy} = ca_x - aa_y + \frac{1}{2}(ab)_y - a_t + 2c_z, \\
(\Phi_\varphi g^{N+})_{zyt} &= (\Phi_\varphi g^{N+})_{zty} = 2bc_y + 2cc_x - b_z - (ac)_y + cb_y, & (\Phi_\varphi g^{N+})_{zzz} &= ab_z, \\
(\Phi_\varphi g^{N+})_{zzt} &= (\Phi_\varphi g^{N+})_{ztz} = \frac{1}{2}c(ab)_x - \frac{1}{2}(a+b)a_t - \frac{1}{4}(a-b)(ab)_y + (bc)_z + ac_z, & (12) \\
(\Phi_\varphi g^{N+})_{ztt} &= 2c(bc)_x + 4cc_z - (a-b)(bc)_y - 2ca_t - bb_z, & (\Phi_\varphi g^{N+})_{txx} &= -4c_x, \\
(\Phi_\varphi g^{N+})_{txy} &= (\Phi_\varphi g^{N+})_{tyx} = b_x - 2c_y - a_x, & (\Phi_\varphi g^{N+})_{txz} &= (\Phi_\varphi g^{N+})_{tzx} = a_t - ac_x - 2c_z, \\
(\Phi_\varphi g^{N+})_{txt} &= (\Phi_\varphi g^{N+})_{ttx} = bb_x - \frac{1}{2}(ab)_x - 2cc_x - cb_y + b_z, \\
(\Phi_\varphi g^{N+})_{tyy} &= 2(b_y - a_y), & (\Phi_\varphi g^{N+})_{tyz} &= (\Phi_\varphi g^{N+})_{tzy} = b_z - \frac{1}{2}(a-b)a_x - (ac)_y, \\
(\Phi_\varphi g^{N+})_{tyt} &= (\Phi_\varphi g^{N+})_{tty} = (b-a)c_x - 4cc_y + 2c_z - a_t - \frac{1}{2}b(a-b)_y, \\
(\Phi_\varphi g^{N+})_{tzz} &= aa_t - 2ac_z, & (\Phi_\varphi g^{N+})_{ttt} &= (b-a)(bc)_x - 2c(bc)_y + 2bc_z - ba_t, \\
(\Phi_\varphi g^{N+})_{tzt} &= (\Phi_\varphi g^{N+})_{ttz} = -\frac{1}{4}(a-b)(ab)_x - \frac{1}{2}c(ab)_y + ca_t - 2cc_z + \frac{1}{2}(a+b)b_z.
\end{aligned}$$

From these equations we have

Theorem 2. *The triple (M, φ, g^{N+}) is Kähler–Norden–Walker if and only if the following PDEs hold:*

$$a_x = a_y = c_x = c_y = b_x = b_y = b_z = 0, \quad a_t - 2c_z = 0. \quad (13)$$

Example. Let $c = 0$ (for Walker metrics ${}^w g$ with $c = 0$, see [10]). Then the triple (M, φ, g^{N+}) with metric

$$g^{N+} = \begin{pmatrix} 0 & -2 & 0 & -b(t) \\ -2 & 0 & -a(z) & 0 \\ 0 & -a(z) & 0 & \frac{1}{2}(1 - a(z)b(t)) \\ -b(t) & 0 & \frac{1}{2}(1 - a(z)b(t)) & 0 \end{pmatrix}$$

is always Kähler–Norden–Walker.

Let (M, φ, g) be an almost Hermitian manifold. The Goldberg conjecture [11, 12] states that an almost Hermitian manifold (M, φ, g) must be Kähler (or φ must be integrable) if the following three conditions are imposed: (G_1) if M is compact and (G_2) g is Einstein, and (G_3) if the fundamental 2-form is closed. Despite many papers by various authors concerning the Goldberg conjecture, there are only two papers by Sekigawa [13, 14] which obtained substantial results to the original Goldberg conjecture. Let (M, φ, g) be an almost Hermitian manifold, which satisfies the three conditions $(G_1), (G_2)$ and (G_3) . If the scalar curvature of M is nonnegative, then φ must be integrable.

Let now $(M, \varphi, {}^w g)$ be an Hermitian–Walker manifold with the proper almost complex structure φ and the metric ${}^w g$ (see (1)). From Theorem 1, we have

Theorem 3. *Let $(M, \varphi, {}^w g)$ be an Hermitian – Walker manifold with the proper almost complex structure φ . The proper almost complex structure φ on a Walker manifold $(M, {}^w g)$ is integrable if $\phi_\varphi g^{N+} = 0$, where g^{N+} is a Norden – Walker metric defined by (3).*

2. Twin Norden – Walker metrics

Let (M, φ, g^{N+}) be an almost Norden – Walker manifold. The associated Norden – Walker metric of almost Norden – Walker manifold is defined by

$$G(X, Y) = (g^{N+} \circ \varphi)(X, Y) \quad (14)$$

for all vector fields X and Y on M . One can easily prove that G is a Norden – Walker metric G^{N+} (see (4)), which is called the twin metric of g^{N+} and it plays a role similar to the Kähler form in Hermitian Geometry. We shall now apply the ϕ_φ -operator to the pure metric G^{N+} :

$$\begin{aligned} (\phi_\varphi G^{N+})(X, Y, Z) &= (\varphi X)(G^{N+}(Y, Z)) - X(G^{N+}(\varphi Y, Z)) + \\ &+ G^{N+}((L_Y \varphi)X, Z) + G^{N+}(Y, (L_Z \varphi)X) = \\ &= (L_{\varphi X} G^{N+} - L_X(G^{N+} \circ \varphi))(Y, Z) + \\ &+ G^{N+}(Y, \varphi L_X Z) - G^{N+}(\varphi Y, L_X Z) = \\ &= (\phi_\varphi G^{N+})(X, \varphi Y, Z) + G^{N+}(N_\varphi(X, Y), Z). \end{aligned} \quad (15)$$

Thus (15) implies the following

Theorem 4. *In an almost Norden – Walker manifold (M, φ, g^{N+}) , we have*

$$\phi_\varphi G^{N+} = (\phi_\varphi g^{N+}) \circ \varphi + g^{N+} \circ (N_\varphi).$$

Corollary 1. *In a Norden – Walker manifold (M, φ, g^{N+}) the following conditions are equivalent:*

- a) $\phi_\varphi g^{N+} = 0$,
- b) $\phi_\varphi G^{N+} = 0$.

We denote by $\nabla_{g^{N+}}$ the covariant differentiation of Levi – Civita connection of Norden metric g^{N+} . Then we have

$$\nabla_{g^{N+}} G^{N+} = (\nabla_{g^{N+}} g^{N+}) \circ \varphi + g^{N+} \circ (\nabla_{g^{N+}} \varphi) = g^{N+} \circ (\nabla_{g^{N+}} \varphi),$$

which implies $\nabla_{g^{N+}} G^{N+} = 0$ by virtue of Theorem 1 ($\nabla_{g^{N+}} \varphi = 0$). Therefore, we have

Theorem 5. *Let (M, φ, g^{N+}) be a Kähler – Norden – Walker manifold. Then the Levi – Civita connection of Norden – Walker metric g^{N+} coincides with the Levi – Civita connection of twin Norden – Walker metric G^{N+} .*

3. Quasi-Kähler – Norden – Walker manifolds

The basis class of non-integrable almost complex manifolds with Norden metric is the class of the quasi-Kähler manifolds. An almost Norden manifold (M, φ, g^{N+}) is called quasi-Kähler [15] if

$$\sigma_{X,Y,Z} g^{N+}((\nabla_X \varphi)Y, Z) = 0,$$

where σ is the cyclic sum by three arguments.

By setting $(L_Y\varphi)X = L_Y(\varphi X) - \varphi(L_Y X) = \nabla_Y(\varphi X) - \nabla_{\varphi X}Y - \varphi(\nabla_Y X) + \varphi(\nabla_X Y)$ and using (8), we see that $(\Phi_\varphi g^{N^+})(X, Y, Z)$ may be expressed as

$$(\Phi_\varphi g^{N^+})(X, Y, Z) = -g^{N^+}((\nabla_X \varphi)Y, Z) + g^{N^+}((\nabla_Y \varphi)Z, X) + g^{N^+}((\nabla_Z \varphi)X, Y).$$

If we add $(\Phi_\varphi g^{N^+})(X, Y, Z)$ and $(\Phi_\varphi g^{N^+})(Z, Y, X)$, then by virtue of $g^{N^+}(Z, (\nabla_Y \varphi)X) = g^{N^+}((\nabla_Y \varphi)Z, X)$, we find

$$(\Phi_\varphi g^{N^+})(X, Y, Z) + (\Phi_\varphi g^{N^+})(Z, Y, X) = 2g^{N^+}((\nabla_Y \varphi)Z, X).$$

Since $(\Phi_\varphi g^{N^+})(X, Y, Z) = (\Phi_\varphi g^{N^+})(X, Z, Y)$, from last equation we have

$$(\Phi_\varphi g^{N^+})(X, Y, Z) + (\Phi_\varphi g^{N^+})(Y, Z, X) + (\Phi_\varphi g^{N^+})(Z, X, Y) = \sigma_{X,Y,Z} g^{N^+}((\nabla_X \varphi)Y, Z).$$

Thus we have

Theorem 6. *Let (M, φ, g^{N^+}) be an almost Norden–Walker manifold. Then the Norden–Walker metric g^{N^+} is quasi-Kähler–Norden–Walker if and only if*

$$(\Phi_\varphi g^{N^+})(X, Y, Z) + (\Phi_\varphi g^{N^+})(Y, Z, X) + (\Phi_\varphi g^{N^+})(Z, X, Y) = 0 \quad (16)$$

for any $X, Y, Z \in \mathfrak{S}_0^1(M_{2n})$.

From (1.) and (16) we have

Theorem 7. *A triple (M, φ, g^{N^+}) is a quasi-Kähler–Norden–Walker manifold if and only if the following PDEs hold:*

$$b_x = b_y = b_z = 0, a_y - 2c_x = 0, a_x + 2c_y = 0, (b - a)c_x - 2cc_y + 2c_z - a_t = 0.$$

We thank Professor Yasuo Matsushita for valuable comments. This paper is supported by The Scientific and Technological Research Council of Turkey (TBAG-108T590).

Резюме

A.A. Салимов, М. Исчан, С. Туранли Дифференциальная геометрия многообразий Уокера.

В статье рассматриваются интегрируемость и голоморфность структуры Нордена–Уокера (M, g^{N^+}, φ) , а также дается характеристика метрики Кэлера–Нордена–Уокера g^{N^+} .

Ключевые слова: структура Нордена–Уокера, многообразие Уокера, чистое тензорное поле, метрика Кэлера–Нордена–Уокера, голоморфное тензорное поле, двойная метрика.

References

1. Walker A.G. Canonical form for a Rimanian space with a parallel field of null planes // Quart. J. Math. Oxford. – 1950. – V. 1, No 2. – P. 69–79.
2. Matsushita Y. Walker 4-manifolds with proper almost complex structure // J. Geom. Phys. – 2005. – V. 55. – P. 385–398.
3. Bonome A., Castro R., Hervella L.M., Matsushita Y., Construction of Norden structures on neutral 4-manifolds // JP J. Geom. Topol. – 2005. – V. 5, No 2. – P. 121–140.

4. *Salimov A.A., Iscan M., Akbulut K.* Some remarks concerning hyperholomorphic B-manifolds // Chin. Ann. Math. Ser. B. – 2008. – V. 29, No 6. – P. 631–640.
5. *Ganchev G.T., Borisov A.V.* Note on the almost complex manifolds with a Norden metric // Compt. Rend. Acad. Bulg. Sci. – 1986. – V. 39, No 5. – P. 31–34.
6. *Iscan M., Salimov A.A.* On Kähler-Norden manifolds // Proc. Indian Acad. Sci. Math. Sci. – 2009. – V. 119, No 1. – P. 71–80.
7. *Norden A.P.* On a certain class of four-dimensional A-spaces // Izvestiya VUZ. Matematika. – 1960. – No 4. – P. 145–157 [in Russian].
8. *Salimov A.A., Iscan M., Etayo F.* Paraholomorphic B-manifold and its properties // Topol. Appl. – 2007. – V. 154, No 4. – P. 925–933.
9. *Kruchkovich G.I.* Hypercomplex structure on a manifold I // Trudy Seminara Vect. Tens. Anal. – M.: Moscow Univ., 1972. – No 16. – P. 174–201 [in Russian].
10. *Matsushita Y.* Four-dimensional Walker metrics and symplectic structure // J. Geom. Phys. – 2004. – V. 52, No 1. – P. 89–99.
11. *Goldberg S.I.* Integrability of almost Kahler manifolds // Proc. Amer. Math. Soc. – 1969. – V. 21. – P. 96–100.
12. *Matsushita Y.* Counterexamples of compact type to the Goldberg conjecture and various version of the conjecture // Proc. of 8th Int. Workshop on Complex Structures and Vector Fields / Ed. by S. Dimiev, K. Sekigawa. – World Scientific Publ., 2007. – P. 222–233.
13. *Sekigawa K.* On some 4-dimensional compact Einstein almost Kähler manifolds // Math. Ann. – 1985. – V. 271. – P. 333–337.
14. *Sekigawa K.* On some compact Einstein almost Kähler manifolds // J. Math. Soc. Japan. – 1987. – V. 36. – P. 677–684.
15. *Manev M., Mekerov D.* On Lie groups as quasi-Kähler manifolds with Killing Norden metric // Adv. Geom. – 2008. – V. 8, No 3. – P. 343–352.
16. *Salimov A.A., Iscan M.* Some properties of Norden – Walker metrics // Kodai Math. J. – 2010. – V. 33. – P. 283–293.

Поступила в редакцию
27.10.10

Salimov, Arif A. – Doctor of Science, Professor, Department of Mathematics, Faculty of Sciences, Atatürk University, Erzurum, Turkey.

Салимов, Ариф Агаджан оглы – доктор физико-математических наук, профессор отделения математики факультета естественных наук Университета Ататюрка, г. Эрзурум, Турция.

E-mail: asalimov@atauni.edu.tr

Iscan, Murat – PhD, Assistant Professor, Department of Mathematics, Faculty of Sciences, Atatürk University, Erzurum, Turkey.

Исчан, Мурат – доктор наук, и.о. доцента отделения математики факультета естественных наук Университета Ататюрка, г. Эрзурум, Турция.

E-mail: miscan@atauni.edu.tr

Turanli, Sibel – Research Assistant, Department of Mathematics, Faculty of Sciences, Atatürk University, Erzurum, Turkey.

Туранли, Сибел – аспирант отделения математики факультета естественных наук Университета Ататюрка, г. Эрзурум, Турция.

E-mail: sibelturanli@hotmail.com