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AFFINE TRANSFORMATIONS OF MANIFOLDS WITH LINEAR CONNECTION AND AUTOMORPHISMS OF LINEAR ALGEBRAS

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In this paper, we show that the group of affine transformations of a manifold M_n with linear connection ∇ induces in each tangent space $T_{x_0}(M_n)$ of M_n a group of automorphisms of the linear algebra $\mathbf{A}$ with operations generated by the covariant differentials of the torsion and curvature tensor fields of ∇. The study of the automorphism group of this algebra allows us to estimate the dimension of the group of affine transformations of the space (M_n, ∇).

1. Affine transformations of manifolds with linear connection

Let M_n be a connected smooth manifold of class C^∞, ∇ a linear connection on M_n, $L(M_n)$ the linear frame bundle of M_n. For a vector field X on M_n, we denote by $X^{(0)}$ the natural lift of X to $L(M_n)$. Recall the following

Definition 1.1. A diffeomorphism $f : M_n \to M_n$ is said to be affine if

$$df(\nabla_X Y) = \nabla_{dfX}\, dfY$$

for any vector fields X and Y on M_n.

In this definition, df denotes the differential of f.

As is known, the set G of all affine transformations of (M_n, ∇) is a Lie group whose dimension is not greater than $n^2 + n$, where n is the dimension of M_n ([1], p. 215).

Denote by $g(M_n)$ the Lie algebra of infinitesimal transformations of the group G of affine transformations of M_n. For each $X \in g(M_n)$, we have

$$L_X \nabla = 0,$$

where L_X denotes the Lie derivative with respect to a vector field X.

With every linear connection, there are naturally associated two tensor fields: the torsion tensor field T and the curvature tensor field R which are defined, respectively, by the identities

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Let K be a tensor field of type (r, s), where $r = 0$ or 1.

Definition 1.2. A tensor field K is said to be invariant with respect to a diffeomorphism f if

$$df(K(X_1, X_2, \dots, X_s)) = K(dfX_1, dfX_2, \dots, dfX_s).$$

for any vector fields $X_1, X_2, \dots, X_s$ on M_n.

One can easily check that the tensor fields T and R are invariant with respect to affine transformations of (M_n, ∇).
