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Components of Functions Summable by Elements of Families of Splash Functions

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In this paper we consider representation of functions by elements of a family of splash functions in the form

$$\varphi_{k,j}(t) = 2^{nk/2}\varphi(2^k t - j), \quad j \in \mathbb{Z}^n, \quad k = 0, 1, \dots \tag{1}$$

In order to obtain the corresponding representation theorems, we establish correlations for components of functions summable by elements of family (1):

$$(f, \varphi_{k,j}) = \int_{\mathbb{R}^n} f(t)\varphi_{k,j}(t)\,dt, \quad j \in \mathbb{Z}^n, \quad k = 0, 1, \dots \tag{2}$$

The main result of the paper (Theorem 2) implies the representation theorem stated as a hypothesis in [1] (P. 28, remark 5).

Theorem 1. *Let a function $\varphi \in L_1 \cap L_p(\mathbb{R}^n)$, $1 < p < \infty$, satisfy the following condition: A constant M exists such that any numerical family $\{c_j\} \in \ell_p(\mathbb{Z}^n)$ meets the inequality*

$$\Big\| \sum_{j\in\mathbb{Z}^n} c_j \varphi(\cdot - j) \Big\|_p \le M \Big(\sum_{j\in\mathbb{Z}^n} |c_j|^p \Big)^{1/p}. \tag{3}$$

Then any function $f \in L_q(\mathbb{R}^n)$, $1/p + 1/q = 1$, satisfies the correlation

$$\lim_{k\to\infty} 2^{nk(1/2-1/q)} \Big(\sum_{j\in\mathbb{Z}^n} |(f, \varphi_{j,k})|^q \Big)^{1/q} = \Big| \int_{\mathbb{R}^n} \varphi(t)\,dt \Big| \|f\|_q. \tag{4}$$

Proof. Let us first assume that the function f is continuous and has a compact support. For a vector $x = (x_1, \dots, x_n)$ we put $[x] = ([x_1], \dots, [x_n])$, where $[x_\nu]$ is the integer part of the value x_ν, $1 \le \nu \le n$. Consider the functions

$$f_k(x) = \int_{\mathbb{R}^n} f(t) 2^{nk} \varphi(2^k t - [2^k x])\,dt, \quad k = 0, 1, \dots$$

Let us prove that the sequence of functions $f_k(x)$, $k = 0, 1, \dots$, converges to the function $\int_{\mathbb{R}^n} \varphi(t)\,dt\, f(x)$ in the metric of the space $L_q(\mathbb{R}^n)$. We have

$$f_k(x) - \int_{\mathbb{R}^n} \varphi(t)\,dt\, f(x) = \int_{\mathbb{R}^n} f(t) 2^{nk} \varphi(2^k t - [2^k x])\,dt - \int_{\mathbb{R}^n} f(x) 2^{nk}\varphi(2^k t)\,dt$$
$$= \int_{\mathbb{R}^n} (f(2^{-k}[2^k x] + t) - f(x)) 2^{nk} \varphi(2^k t)\,dt.$$

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