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The Operad of Finite Labeled Lattices

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This paper is a continuation of the series of papers [1–5]. We study various classes of finite labeled graphs (nonoriented or oriented) which form operads with respect to the composition operations introduced in [1].

The partially ordered sets and, in particular, lattices, form important classes of oriented graphs. The families of finite labeled partially ordered sets and lattices also possess operad structures. In this paper, we study elements of the operad of lattices that admit no nontrivial decomposition into operad composition. It turns out that the question on such lattices is closely related to the question on the structure of the congruence $\Theta(a, b)$ from [6]. Namely, an element of the operad of finite labeled lattices (i.e., a finite lattice) is operad indecomposable if and only if, on this lattice, there exist no congruences of the following form. Elements $a \leq b$ are fixed (where either $a \neq 0$ or $b \neq 1$), and $x \sim y$ if (and only if) either $a \leq x, y \leq b$ or $x = y$. We succeeded in finding many examples of lattices with this property.

Note that the operad composition on the family of finite partially ordered sets bears similarities to Plonka's Agassiz sums (see the definition in [7], P. 46; [8], P. 314–315).

It should be also noted that in the literature on the graph theory (independently of the literature on the universal algebra) one can find constructions similar to the operad composition from [1] (see, e.g., [9, 10]). The operad language, however, is not used.

In this paper, we construct the operad of finite labeled lattices and its generalization, the operad of finite labeled partially ordered sets. We establish a criterion of operad decomposability for lattices. Using this criterion, we prove operad indecomposability of a series of families of lattices. In particular, the following classes of lattice turn out to be indecomposable: simple lattices, lattices decomposable into product, lattices with relative complements, modular lattices with complements, many of finite distributive lattices, and all finite geometric lattices.

We use definitions, results, and notation from [1] with changes appeared later in [11]. Some preliminary versions of results of this paper were announced in [3–5].

Denote by $\mathrm{Lat}(n)$ the set of finite lattices with n elements enumerated by the numbers from 1 to n. We consider the same lattice with different enumerations of elements as different elements of $\mathrm{Lat}(n)$. The group Σ_n of permutations of degree n acts on $\mathrm{Lat}(n)$ by permutation of numbers of elements. From the aforesaid it follows that the orbits of the action of Σ_n correspond to different lattices. We introduce an operad structure on the family $\mathrm{Lat} = \{\mathrm{Lat}(n) \mid n \geq 1\}$ regarding lattices as a partial case of oriented graphs: $x \leq y$ means that there is a unique arc from a vertex x to a vertex y. Then the definition of the composition for oriented graphs (with regard to the structure of partially ordered sets) takes the form

$$\mathrm{Lat}(m) \times \mathrm{Lat}(n_1) \times \cdots \times \mathrm{Lat}(n_m) \to \mathrm{Lat}(n_1 + \cdots + n_m), \quad (L_0, L_1, \ldots, L_m) \mapsto L_0 L_1 \ldots L_m.$$

Elements of $L = L_0 L_1 \ldots L_m$ are disjoint unions of elements of $L_1, \ldots, L_m$. In addition, the following renumbering takes place: the kth element of L_i gets the number $n_1 + \cdots + n_{k-1} + k$, where $n_j = |L_j|$, and the following inequality relations hold:

1) for $x, y \in L_i$, we have $x \leq y$ in $L_0 L_1 \ldots L_m$ if and only if $x \leq y$ in L_i;

2) for $x \in L_i$, $y \in L_j$, $i \neq j$, we have $x \leq y$ in $L_0 L_1 \ldots L_m$ if and only if $i \leq j$ in L_0.

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