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The Operad of Finite Labeled Tournaments

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Abstract—We study the operad of finite labeled tournaments. We describe the structure of suboperads of this operad generated by simple tournaments. We prove that a suboperad generated by a tournament with two vertices (i.e., the operad of finite linearly ordered sets) is isomorphic to the operad of symmetric groups, and a suboperad generated by a simple tournament with more that two vertices is isomorphic to the quotient operad of the free operad with respect to a certain congruence. We obtain this congruence explicitly.

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In [1] one describes two ways for defining an operad on various classes of finite labeled graphs, both nonoriented and oriented ones. These operads have many interesting examples of suboperads. In this paper we study one of suboperads of the operad of finite oriented graphs (orgraphs), namely, the operad of tournaments. Tournaments (or robin tournaments) constitute a well-known class of oriented graphs; it is being thoroughly studied (e.g., [2]). However, tournaments were not considered yet in terms of operads. At the same time, a construction similar to the operad composition of graphs [1] is well-known in the theory of tournaments (e.g., [3] and [2]).

This paper consists of two Sections. In the first Section we recall the definitions of some known operads, in particular, the operad Dir of finite oriented labeled graphs and some its suboperads. We prove, in particular, that tournaments form a suboperad. A suboperad of the operad of tournaments is the operad LOS of linearly ordered finite labeled sets. The main result of Section 1 is the construction of the isomorphism between the operad LOS and the operad Σ of symmetric groups (see [4] for the detailed description of the latter operad). In Section 2 we study suboperads of the operad of tournaments generated by arbitrary simple tournaments with no less than three vertices. The following assertion is the main result of Section 2. Each mentioned suboperad is isomorphic to the quotient operad of the free operad with one-element basis with respect to an explicitly described congruence defined by a group of automorphisms of a given simple tournament. In particular, if this group is trivial, then the operad generated by this simple tournament is free. Note that one can also treat the operad LOS as a suboperad of the operad of tournaments generated by a simple tournament with two vertices and thus describe suboperads generated by all simple tournaments.

Since one can represent any tournament as the operad composition of simple tournaments (accurate to the enumeration of vertices), the following natural question arises: Is it true that the operad of all tournaments is a categorical coproduct (in the category of symmetric operads) of the operad LOS and all suboperads generated by simple tournaments with no less than three vertices? If so, then the family of all tournaments admits a sufficiently transparent (from the algebraic point of view) description.

In this paper we use results of the theory of operads (see [1, 5–7], and references therein). Denotations mainly correspond to those of paper [5]. Theorem 3 and a particular case of Theorem 4 adduced in this paper were announced in [8].

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