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# AN ANALOG OF THE FRAENKEL PROBLEM FOR EQUATION OF THE SECOND KIND

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## 1. Statement of problem

Let us consider the equation

$$u_{xx} + yu_{yy} + (-n + 1/2)u_y = 0, \quad n \in N, \tag{1}$$

in a mixed domain $D$, whose elliptic part $D_1$ coincides with the whole upper halfplane and the hyperbolic part consists of the two infinite triangles: $D_2$ bounded by the characteristics $y = 0$ and $x - 2\sqrt{-y} = 0$, and $D_3$ bounded by the characteristics $y = 0$ and $x + 2\sqrt{-y} = 0$.

In [1]–[4] problems of the Tricomi and Bitsadze–Samarskiĭ types were investigated for equation (1) in the case of unbounded domains. In this article we consider a certain analog the Fraenkel problem.

**Problem** $F$. *In the domain $D$ find a function $u(x,y)$ with the properties*

1) *$u(x,y)$ belongs to $C(D_1 \cup \{y = 0\}) \cap C(D_2 \cup \{y = 0\} \cup \{x - 2\sqrt{-y} = 0\}) \cap C(D_3 \cup \{y = 0\} \cup \{x + 2\sqrt{-y} = 0\})$;*
2) *the relations are valid*

$$u = o(R^{2n+1}), \quad u_x = o(R^{2n}), \quad u_y = o(R^{2n-1}) \tag{2}$$

*as $R \to +\infty$, where $R^2 = x^2 + 4y$, $(x,y) \in D_1$;*

3) *$u(x,y)$ belongs to $C^2(D_1 \cup D_2 \cup D_3)$ and satisfies equation* (1) *in $D_1 \cup D_2 \cup D_3$;*
4) *the limits exist $(x \neq 0)$*

$$\nu_i(x) = \lim_{y \to 0,\ (x,y) \in D_i} |y|^{-n+1/2}[u(x,y) - A(x,y,\tau_i)]_y, \quad i = 1,2,3, \tag{3}$$

*and the equalities are fulfilled*

$$\nu_1(x) = (-1)^n \nu_2(x), \quad x > 0, \tag{4}$$

$$\nu_1(x) = 0, \quad \nu_3(x) = 0, \quad x < 0, \tag{5}$$

*where*

$$\tau_i(x) = \lim_{y \to 0,\ (x,y) \in D_i} u(x,y), \tag{6}$$

$$A(x,y,\tau) = \sum_{s=1}^{n} \frac{\tau^{(2s)}(x)(-1)^s}{(-n+1/2)_s s!} y^s,$$

$(\alpha)_0 = 1$, $(\alpha)_s = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+s-1)$;

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5) *$u(x,y)$ satisfies the boundary value condition* $(y<0)$

$$u(x,y)|_{x=2\sqrt{-y}} = u(x,y)|_{x=-2\sqrt{-y}}; \tag{7}$$

6) *the gluing conditions are fulfilled*

$$\tau_1(x) = \tau_2(x), \qquad x \ge 0, \tag{8}$$
$$\tau_1(x) = \tau_3(x) + g(x), \quad x \le 0, \tag{9}$$

*where $g(x)$ is a given function*;

7) *the equalities are fulfilled*

$$\tau_i^{(s)}(0) = 0, \quad s = \overline{0,n}. \tag{10}$$

The following condition is assumed to be fulfilled:

**Condition 1.** The function $g(x)$ is in $C^n(-\infty,0] \cap C^{2n+1,\gamma}(-\infty,0)$, $\gamma > 0$,

$$g^{(s)}(0) = 0, \quad s = \overline{0,n},$$

the derivative $g^{(n+1)}(x)$ can possess a singularity for $x=0$ of order less than one and must possess the representation $g^{(n+1)}(x) = O(|x|^{-\beta})$ as $x \to \infty$, where $\beta > 1/2$.

A solution will be sought in the class of functions for which $\tau_i(x)$ satisfy conditions analogous to Condition 1 with regard for the domain of definition, while for $\nu_i(x)$ the following condition holds:

**Condition 2.** The functions $\nu_i(x)$ are continuous in the respective domains of definition, can possess a singularity for $x=0$ of order which is less that $n+1$, and are bounded on infinity.

## 2. The basic relation from elliptic subdomain

To solve the problem by the method of integral equations we will need basic relations between $\tau$ and $\nu$. In the elliptic subdomain we use the solution of the Dirichlet problem with the boundary value condition (6) for $i=1$ (see [5], p. 588)

$$u(x,y) = \frac{n!2^{2n+1}}{\sqrt{\pi}\Gamma(n+1/2)} y^{n+1/2} \int_{-\infty}^{+\infty} \tau_i(\xi)[(x-\xi)^2 + 4y]^{-n-1} d\xi. \tag{11}$$

One can easily verify that this formula is valid in the case where the given function possesses on infinity a singularity of an order lesser than $2n+1$, and the resulting function (11) will satisfy conditions (2). By substituting representation (11) into (3), in a way analogous to that in [3] we prove that the following lemma takes place.

**Lemma 1.** *The basic relation from the elliptic subdomain has the form*

$$\Gamma(n+1/2)\nu_1(x) = \frac{n!(n+1/2)2^{2n+1}}{\sqrt{\pi}(2n+1)!} \frac{d^n}{dx^n} \int_{-\infty}^{+\infty} \frac{\tau_1^{(n+1)}(\xi)}{\xi - x} d\xi. \tag{12}$$

## 3. Basic relation from hyperbolic domains

Let us use the solution of the Cauchy type problem with the initial conditions (6), (3) for $i = 2, 3$, which in the corresponding domains have the form

$$u(x,y) = \frac{1}{2}\sum_{s=0}^{n} \frac{n!(2n-s)!2^{2s}}{s!(n-s)!(2n)!}(-y)^{s/2}[\tau_i^{(s)}(x-2\sqrt{-y}) + (-1)^s\tau_i^{(s)}(x+2\sqrt{-y})] - \\ - \frac{2(2n+1)!}{(n+1)!^2}(-y)^{n+1/2}\int_0^1 \nu_i(\zeta)[\xi(1-\xi)]^n d\xi, \quad (13)$$

where $\zeta = x - 2\sqrt{-y}(1-2\xi)$.

Condition (7) can be written in the form

$$u\Big(x, -\frac{x^2}{4}\Big) = u\Big(-x, -\frac{x^2}{4}\Big). \quad (14)$$

By substituting (13) into (14) in a way similar to that in [1] one can prove

**Lemma 2.** *The basic relation from the hyperbolic subdomains has the form*

$$\Gamma(n+1/2)\nu_2(x) - \Gamma(-n+1/2)\tau_2^{(2n+1)}(x) = \Gamma(n+1/2)\nu_3(-x) + \Gamma(-n+1/2)\tau_3^{(2n+1)}(-x). \quad (15)$$

## 4. Derivation of the integral equations and solution of Problem $F$

From relations (12), (15) with regard for conditions (4), (5), (8), (9) and the equality

$$\frac{n!(n+1/2)2^{2n+1}(-1)^n}{\sqrt{\pi}(2n+1)!\Gamma(-n+1/2)} = \frac{1}{\pi}$$

we obtain

$$\tau_1^{(2n+1)}(x) + \tau_1^{(2n+1)}(-x) - \frac{1}{\pi}\frac{d^n}{dx^n}\int_{-\infty}^{+\infty}\frac{\tau_1^{(n+1)}(\xi)d\xi}{\xi - x} = g^{(2n+1)}(-x), \quad x > 0, \quad (16)$$

$$\frac{1}{\pi}\frac{d^n}{dx^n}\int_{-\infty}^{+\infty}\frac{\tau_1^{(n+1)}(\xi)d\xi}{\xi + x} = 0, \quad x > 0. \quad (17)$$

In the both relations (16) and (17) we partition the integrals into the two addends: from $-\infty$ to 0 and from 0 to $+\infty$. In the first addends we change the sign of the variation of integration. As a result, equations (16), (17) take the form

$$\tau_1^{(2n+1)}(x) + \tau_1^{(2n+1)}(-x) - \frac{1}{\pi}\frac{d^n}{dx^n}\int_0^{+\infty}\frac{\tau_1^{(n+1)}(\xi)d\xi}{\xi - x} + \frac{1}{\pi}\frac{d^n}{dx^n}\int_0^{+\infty}\frac{\tau_1^{(n+1)}(-\xi)d\xi}{\xi + x} = g^{(2n+1)}(-x), \quad (18)$$

$$\frac{1}{\pi}\frac{d^n}{dx^n}\int_0^{+\infty}\frac{\tau_1^{(n+1)}(\xi)d\xi}{\xi + x} - \frac{1}{\pi}\frac{d^n}{dx^n}\int_0^{+\infty}\frac{\tau_1^{(n+1)}(-\xi)d\xi}{\xi - x} = 0. \quad (19)$$

Equalities (18), (19) represent a system of equations with respect to $\tau_1(x)$ and $\tau_1(-x)$ for $x > 0$. Thus, we have proved the following

**Theorem 1.** *The solution of Problem $F$ can be reduced to solving system of equations* (18), (19).

Let us solve this system. We multiply equation (19) by $(-1)^n$ and add the result to equation (18). As a result, we obtain

$$\frac{d^n}{dx^n}\Big[\rho(x) - \frac{1}{\pi}\int_0^{+\infty}\Big(\frac{1}{\xi - x} - \frac{(-1)^n}{\xi + x}\Big)\rho(\xi)d\xi - (-1)^n g^{(n+1)}(-x)\Big] = 0, \quad (20)$$

where

$$\rho(x) = \tau_1^{(n+1)}(x) + (-1)^n \tau_1^{(n+1)}(-x). \tag{21}$$

Equation (20) is equivalent to the following one:

$$\rho(x) - \frac{1}{\pi}\int_0^{+\infty}\left(\frac{1}{\xi - x} - \frac{(-1)^n}{\xi + x}\right)\rho(\xi)d\xi = (-1)^n g^{(n+1)}(-x) + \sum_{s=0}^{n-1} c_s x^s,$$

where $c_s$ are arbitrary constants.

Let us realize the change of variables $\xi^2 = \sigma$, $x^2 = \eta$. For $n$ odd we have

$$z(\eta) - \frac{1}{\pi}\int_0^{+\infty}\frac{z(\sigma)d\sigma}{\sigma - \eta} = f(\eta) + \sum_{s=0}^{n-1} c_s \eta^{s/2}, \tag{22}$$
$$z(\eta) = \rho(x), \quad f(\eta) = (-1)^n g^{(n+1)}(-x);$$

while with $n$ even we have

$$z(\eta) - \frac{1}{\pi}\int_0^{+\infty}\frac{z(\sigma)d\sigma}{\sigma - \eta} = f(\eta) + \sum_{s=0}^{n-1} c_s \eta^{(s-1)/2}, \tag{23}$$
$$z(\eta) = \rho(x)/x, \quad f(\eta) = (-1)^n g^{(n+1)}(-x)/x.$$

Let us estimate the behavior of the functions on the ends of the interval of integration. From Condition 1 it follows that, if $n$ is odd, then the function $f(\eta)$ can have a singularity for $\eta = 0$ of order lesser than 1/2 and as $\eta \to +\infty$ it must have a zero of order exceeding 1/4; if $n$ is even, then the function $f(\eta)$ can have a singularity for $\eta = 0$ of order lesser than 1 and as $\eta \to +\infty$ it must have a zero of order exceeding 3/4. The function $z(\eta)$ is assumed to possess the same behavior.

Equations of the form (22), (23) are solvable when the right-hand side vanishes at infinity (see [6]). Hence we have the equalities $c_s = 0$, $s = \overline{0, n-1}$ in case of equation (22) and $c_s = 0$, $s = \overline{1, n-1}$ in case of equation (23). Now in these equations we realize the substitution (see [6])

$$\eta = \zeta/(1-\zeta), \quad \sigma = t/(1-t),$$
$$z(\eta) = v(\zeta)(1-\zeta), \quad f(\eta) = b(\zeta)(1-\zeta).$$

These equations take the form

$$v(\zeta) - \frac{1}{\pi}\int_0^1 \frac{v(t)dt}{t - \zeta} = b(\zeta) + \frac{c_0}{\sqrt{\zeta(1-\zeta)}}; \tag{24}$$

besides, $c_0 = 0$ for odd $n$.

Let us note that solutions of an equation of the form (24), which are unbounded for $\zeta = 0$, possess a singularity of order 1/4, while being unbounded for $\zeta = 1$ possess a singularity of order 3/4 at the indicated point.

Let $n$ be odd. The function $b(\zeta)$ can possess singularities for $\zeta = 0$ of order lesser that 1/2, for $\zeta = 1$ of order lesser that 3/4. Singularities of that kind are admissible for the solution $v(\zeta)$. Therefore it is necessary to apply the formula of a solution bounded for $\zeta = 1$.

Let $n$ be even. The function $b(\zeta)$ can possess singularities for $\zeta = 0$ of order lesser than 1, for $\zeta = 1$ of order lesser that 1/4. Consequently, it is necessary to apply again the formula of the solution bounded for $\zeta = 1$. Moreover, the relation must hold $c_0 = 0$. Otherwise the function $v(\zeta)$ will have with $\zeta = 1$ a singularity of order 1/2, which exceeds the admitted value.

The above solution of unique. Hence we determine $v(\zeta)$, and consequently, also the function $\rho(x)$. Now, when the function $\rho(x)$ has been found, from formula (21) we express

$$\tau_1^{(n+1)}(-x) = (-1)^n \rho(x) - (-1)^n \tau_1^{(n+1)}(x) \tag{25}$$

and substitute the result into equation (19). This gives us

$$\frac{d^n}{dx^n}\left[\frac{1}{\pi}\int_0^{+\infty}\left(\frac{1}{\xi-x}+\frac{(-1)^n}{\xi+x}\right)\tau_1^{(n+1)}(\xi)d\xi-\frac{1}{\pi}\int_0^{+\infty}\frac{\rho(\xi)}{\xi-x}d\xi\right]=0.$$

This equation has the same structure as equation (20). Therefore, after consecutive changes of variables, we obtain

$$\frac{1}{\pi}\int_0^1\frac{p(t)dt}{t-\zeta}=q(\zeta)+\frac{d_0}{\sqrt{\zeta(1-\zeta)}};$$

moreover, $d_0=0$ for $n$ even.

It is admitted that the functions $p(\zeta)$ and $q(\zeta)$ can possess the following singularities: If $n$ is even, then with $\zeta=0$ a singularity of order lesser than $1/2$, for $\zeta=1$ — lesser than $3/4$; if $n$ is odd, then with $\zeta=0$ the order will be lesser than 1, for $\zeta=1$ — lesser than $1/4$.

The solution unbounded for $\zeta=0$ or $\zeta=1$, possesses at such a point a singularity of order $1/2$. Therefore it is necessary to use the formula of the solution bounded for $\zeta=0$ if $n$ is even, and bounded for $\zeta=1$ if $n$ is odd. In addition, from here one can derive that, in the second case, $d_0=0$. The above solutions are unique.

Let us determine $p(\zeta)$, and therefore $\tau_1^{(n+1)}(x)$, $x>0$, as well. Using formula (25), we calculate $\tau_1^{(n+1)}(x)$, $x<0$. Afterwards, with regard for equalities (10), we restore the function $\tau_1(x)$. Then from relations (8), (9), (15) we consecutively determine the functions $\tau_2(x)$, $\tau_3(x)$, and $\nu_2(x)$. If we write the solution in every of subdomains, using formulas (11) and (13), then this solution is unique by virtue of the unique determination of the auxiliary functions and the uniqueness of solutions of the Dirichlet and Cauchy type problems.

Therefore we have proved the following

**Theorem 2.** *Problem F has the unique solution.*

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