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Analysis of Local Dynamics of Difference and Close to Them Differential-Difference Equations

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Abstract—We study the local dynamics of one class of nonlinear difference equations which is important for applications. Using perturbation theory methods, we construct sets of singularly perturbed differential-difference equations that are close (in a sense) to initial difference equations. For the problem on the stability of the zero equilibrium state and for certain infinite-dimensional critical cases, we propose a method that allows us to construct analogs of normal forms. We mean special nonlinear boundary value problems without small parameters, whose nonlocal dynamics describes the structure of solutions to initial equations in a small neighborhood of the equilibrium state. We show that dynamic properties of difference and close to them differential-difference equations considerably differ.

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1. The problem. Difference and differential-difference equations arise in many applications [1, 2], including some models in laser physics, radio electronics, medicine, neural networks, cellular automata, information processing, etc. In this paper, we study equations that play a fundamental role in the theory of difference systems of equations.

Consider one of the simplest scalar nonlinear difference equations in terms of "continuous" time

$$u(t) = au(t-1) + f\,(u(t-1)) \quad (t \in [0, \infty)), \tag{1}$$

where $f(u) = f_2u^2 + f_3u^3 + O(u^4)$. The local dynamics, i.e., the behavior of all solutions to Eq. (1) in a sufficiently small neighborhood of the zero equilibrium state, mainly depends on the coefficient a. Evidently, with $|a| < 1$ all solutions to (1) with a sufficiently small value of $|u_0|$ tend to zero as $n \to \infty$. But with $|a| > 1$ the problem becomes nonlocal. In this case, nonlocal studies are based on works [3–5]. Some results of the local analysis of (1) are given in [6]. Therefore, it makes sense to study only those cases that are close to critical ones in the zero stability problem, when the value of the parameter a is close to ± 1. Let us confine ourselves to studying the most interesting case, when the equality

$$a = -\left(1 + \varepsilon a_1\right), \quad 0 < \varepsilon \ll 1 \tag{2}$$

is valid for some fixed a_1.

Note that under condition (2) for Eq. (1) we observe a case which is close to the critical (infinite-dimensional) one in the problem on the stability of the zero equilibrium state. Indeed, the characteristic equation $1 = -\exp(-\lambda)$ of the linearized (at the zero point) with $\varepsilon = 0$ Eq. (1) has infinitely many roots

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