ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2010, Vol. 54, No. 4, pp. 72–81. © Allerton Press, Inc., 2010.
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# Strong Noncuppability in Low Computably Enumerable Degrees

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Received March 25, 2008

**Abstract**—We prove the existence of noncomputable low computably enumerable degrees $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b}$ is strongly noncuppable to $\mathbf{a}$ in the class $\mathbf{R}$.

**DOI:** 10.3103/S1066369X10040092

Key words and phrases: *computably enumerable sets, Turing degrees, low degrees, noncuppability.*

## 1. INTRODUCTION

In this paper we study the questions of cuppability and noncuppability in the class of computably enumerable (c. e.) degrees $\mathbf{R}$. We prove that there exist noncomputable low c. e. degrees $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b}$ is strongly noncuppable to $\mathbf{a}$ in the class $\mathbf{R}$. Our denotations and terminology are standard, generally they follow Soare [1].

**Definition.** Given Turing degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{0} < \mathbf{b} < \mathbf{a}$, we say that $\mathbf{b}$ *is noncuppable to* $\mathbf{a}$ *in a class of degrees* $\mathcal{C}$, if there is no degree $\mathbf{w} \in \mathcal{C}$ such that $\mathbf{w} < \mathbf{a}$ and $\mathbf{a} = \mathbf{b} \cup \mathbf{w}$. We say that $\mathbf{b}$ *is strongly noncuppable to* $\mathbf{a}$ *in a class of degrees* $\mathcal{C}$, if there is no degree $\mathbf{w} \in \mathcal{C}$ such that $\mathbf{a} \nleq \mathbf{w}$ and $\mathbf{a} \leq \mathbf{b} \cup \mathbf{w}$.

The Cooper and Yates theorem [2] was one of the first results on noncuppability. It states that there exists a noncomputable c. e. degree noncuppable to $\mathbf{0}'$ in the class $\mathbf{R}$. Later Harrington [2] proved that for any high c. e. degree $\mathbf{h}$ there exists a high c. e. degree strongly noncuppable to $\mathbf{h}$ in the class $\mathbf{R}$. Then Cooper [3], Slaman, and Steel [4] proved that there exist noncomputable c. e. degrees $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b}$ is noncuppable to $\mathbf{a}$ in the class of $\Delta^0_2$-degrees.

By studying the cuppability in the class of 2-c. e. degrees $\mathbf{D}_2$ the following theorem was proved.

**Theorem 1.1** (Arslanov [5]). *For any noncomputable* 2*-c. e. degree* $\mathbf{b}$ *there exists a noncomplete* 2*-c. e. degree* $\mathbf{d}$ *such that* $\mathbf{0}' = \mathbf{b} \cup \mathbf{d}$.

As a consequence we obtain that the upper semilattices $\mathbf{R}$ and $\mathbf{D}_2$ are not elementarily equivalent. Theorem 1.1 was generalized by Cooper, Lempp and Watson as follows.

**Theorem 1.2** ([6]). *For any high c. e. degree* $\mathbf{h}$, *for any noncomputable $n$-c. e.* $(n \geq 1)$ *degree* $\mathbf{b} \leq \mathbf{h}$ *there exists a low* 2*-c. e. degree* $\mathbf{d}$ *such that* $\mathbf{h} = \mathbf{b} \cup \mathbf{d}$.

The following sentence shows that classes of low c. e. $\mathbf{R}^{\text{low}}$ and low 2-c. e. $\mathbf{D}_2^{\text{low}}$ degrees are not elementarily equivalent:

$$\varphi = \exists\, \mathbf{a}, \mathbf{b}\, \forall\, \mathbf{w}\ \ [(\mathbf{0} < \mathbf{b} < \mathbf{a}) \wedge [\mathbf{a} \leq \mathbf{w} \vee \mathbf{a} \nleq \mathbf{b} \cup \mathbf{w}]].$$

This sentence can be transformed to the following one without "∪":

$$\psi = \exists\, \mathbf{a}, \mathbf{b}\, \forall\, \mathbf{w}\ \ [(\mathbf{0} < \mathbf{b} < \mathbf{a}) \wedge (\mathbf{a} \leq \mathbf{w} \vee (\exists\, \mathbf{g}\, [\mathbf{a} \nleq \mathbf{g} \wedge \mathbf{b} \leq \mathbf{g} \wedge \mathbf{w} \leq \mathbf{g}]))].$$

Using Theorem 1.2 in the case when $\mathbf{h} = \mathbf{0}'$ and avoiding "∪" we obtain that the sentence $\psi$ is not true in $\mathbf{D}_2^{\text{low}}$, however, it is unknown whether $\psi$ is true in $\mathbf{R}^{\text{low}}$. We will prove the following theorem.

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