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A LOW-RANK APPROXIMATION OF TENSORS AND THE TOPOLOGICAL GROUP STRUCTURE OF INVERTIBLE MATRICES

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Abstract

By a tensor we mean an element of the tensor product of vector spaces over a field. Up to a choice of bases in factors of tensor products, every tensor may be coordinatized, i.e., represented as an array consisting of numbers. The properties of the tensor rank, which is a natural generalization of the matrix rank, have been considered in this paper. The topological group structure of invertible matrices has been studied. The multilinear matrix multiplication has been discussed from the viewpoint of transformation groups. We treat a low-rank tensor approximation in finite-dimensional tensor products. It has been shown that the problem on determining the best rank-n approximation for a tensor of size $n \times n \times 2$ has no solution. To this end, we have used an approximation by matrices with simple spectra.

Keywords: approximation by matrices with simple spectra, group action, low-rank tensor approximation, norm on tensor space, open mapping, simple spectrum of matrix, tensor rank, topological group of invertible matrices, topological transformation group

Introduction

Tensors are ubiquitous in sciences. The subject of tensors is an active research area in mathematics and its applications (see, for example, [1–3] and references therein).

This paper is devoted to the tensor rank and a low-rank approximation of tensors. The tensor rank can be considered as a measure of complexity of tensors. Therefore, one is often required to find an approximation of a given tensor by tensors with lower tensor ranks. In particular, the best low-rank approximation problem for tensors is of great interest in the statistical analysis of multiway data (see, for example, references in [4, p. 1085]). As is known, in general, the best low-rank approximation problem for tensors is ill-posed [4, 5].

A part of motivation for this work comes from our study of the complexity of tensors in homological complexes of Banach spaces [6, 7]. The main part of motivation comes from the results in [4, 8, 9] on tensors in finite-dimensional spaces. In this paper, we consider tensors in finite-dimensional spaces with the Euclidean topology. The properties of these tensors are closely related to the topological group structure of invertible matrices. Here, we deal with the natural topological group action on a space of tensors. We show the ill-posedness of the best rank-n approximation problem in the space of tensors of size $n \times n \times 2$.

1. Tensor rank and its properties

As usual, $\mathbb{N}$ stands for the set of all natural numbers. In the sequel, $l, m, n \in \mathbb{N}$ and $l, m, n \geqslant 2$.

Throughout the note, $\mathbb{F}$ will denote either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. For an element $\mathbf{x} \in \mathbb{F}^l$, we use the notation $\mathbf{x} = (x_1, \dots, x_l)^T$, where $x_i \in \mathbb{F}, i = 1, \dots, l$.

We denote by $\mathbb{F}^{l\times m}$, or by $M_{l,m}(\mathbb{F})$, the linear space of all matrices $A = (a_{ij})$ of size $l \times m$, where $a_{ij} \in \mathbb{F}$, $i = 1, \dots, l$, $j = 1, \dots, m$. The space of all square matrices of order n over the field $\mathbb{F}$ is denoted by $M_n(\mathbb{F})$. The general linear group of degree n, i.e., the group of invertible matrices in $M_n(\mathbb{F})$, is denoted by $GL_n(\mathbb{F})$. The symbol E_n stands for the identity matrix in $M_n(\mathbb{F})$.

For $\mathbf{x} = (x_1, \dots, x_l)^T \in \mathbb{F}^l$ and $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{F}^m$, the matrix $\mathbf{x} \otimes \mathbf{y} \in \mathbb{F}^{l\times m}$ is given by

$$\mathbf{x} \otimes \mathbf{y} = (x_i y_j), \quad \text{where} \quad i = 1, \dots, l, \quad j = 1, \dots, m.$$

Let $\mathbb{F}^{l\times m\times n}$ be the linear space of all arrays $A = (a_{ijk})$ of size $l \times m \times n$, where $a_{ijk} \in \mathbb{F}$, $i = 1, \dots, l$, $j = 1, \dots, m$, $k = 1, \dots, n$. For a tensor $A = (a_{ijk}) \in \mathbb{F}^{l\times m\times n}$, we also use the following notation:

$$A = [A_1|\cdots|A_n],$$

where, for every $r = 1, \dots, n$, the slice A_r is defined by

$$A_r = (a_{ijr}) \in \mathbb{F}^{l\times m}, \quad \text{where} \quad i = 1, \dots, l, \quad j = 1, \dots, m.$$

For $\mathbf{x} = (x_1, \dots, x_l)^T \in \mathbb{F}^l, \mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{F}^m, \mathbf{z} = (z_1, \dots, z_n)^T \in \mathbb{F}^n$,, we define the array $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} \in \mathbb{F}^{l\times m\times n}$ by

$$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} = (x_i y_j z_k), \quad \text{where} \quad i = 1, \dots, l, \quad j = 1, \dots, m, \quad k = 1, \dots, n.$$

Let us consider the bilinear mapping θ and the trilinear mapping τ defined as follows:

$$\theta\colon \mathbb{F}^l \times \mathbb{F}^m \longrightarrow \mathbb{F}^{l\times m} : (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \otimes \mathbf{y};$$

$$\tau\colon \mathbb{F}^l \times \mathbb{F}^m \times \mathbb{F}^n \longrightarrow \mathbb{F}^{l\times m\times n} : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}.$$

It is well known that the pairs $(\mathbb{F}^{l\times m}, \theta)$ and $(\mathbb{F}^{l\times m\times n}, \tau)$ are the tensor products for the corresponding linear spaces. In what follows, elements of the spaces $\mathbb{F}^{l\times m}$ and $\mathbb{F}^{l\times m\times n}$ are called *tensors*.

For the basics of algebraic tensor products, we refer the reader, for instance, to [2, Part I, Ch. 3], [10, Ch. 1], and [11, Ch. 2, § 7].

Both of the l_1-norms on the linear spaces $\mathbb{F}^{m\times n}$ and $\mathbb{F}^{l\times m\times n}$ will be denoted by the same symbol $\|\cdot\|_1$. We recall that the value of the l_1-norm at a tensor A is defined as the sum of absolute values of all entries in A.

All norms on a space of tensors are equivalent and generate the same topology that is called the Euclidean topology. The convergence of a tensor sequence

$$\{A_t\} = \{(a^t_{ijk})\} \subset \mathbb{F}^{l\times m\times n}, \quad t \in \mathbb{N},$$

to a tensor $A = (a_{ijk}) \in \mathbb{F}^{l\times m\times n}$ with respect to this topology is exactly the entrywise convergence, i.e., for every fixed triple of indices $i = 1, \dots, l$, $j = 1, \dots, m$, $k = 1, \dots, n$, one has the equality

$$\lim_{t\to+\infty} a^t_{ijk} = a_{ijk}.$$

In the sequel, we consider spaces of tensors endowed with the Euclidean topologies. The general linear group $GL_n(\mathbb{F})$ is a topological group with respect to that topology.

Definition 1. Tensors $A \in \mathbb{F}^{l\times m}$ and $B \in \mathbb{F}^{l\times m\times n}$ are said to be *elementary* (or *decomposable*) if $A = \mathbf{a}\otimes\mathbf{b}$ and $B = \mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}$ for some vectors $\mathbf{a}, \mathbf{x} \in \mathbb{F}^l$, $\mathbf{b}, \mathbf{y} \in \mathbb{F}^m$ and $\mathbf{z} \in \mathbb{F}^n$.

Definition 2. A tensor $A \in \mathbb{F}^{l\times m}$ or a tensor $B \in \mathbb{F}^{l\times m\times n}$ has *the tensor rank* r if it can be written as a sum of r elementary tensors, but no fewer. We will use the notation $rank(A)$ (or $rank_{\mathbb{F}}(A)$) for the tensor rank of A. Therefore, we may write

$$rank(B) = \min\big\{r \mid B = \sum_{i=1}^{r}\mathbf{x}_i\otimes\mathbf{y}_i\otimes\mathbf{z}_i,\ \text{where } \mathbf{x}_i \in \mathbb{F}^l,\ \mathbf{y}_i \in \mathbb{F}^m,\ \mathbf{z}_i \in \mathbb{F}^n\big\}.$$

As is well known, for $A \in \mathbb{F}^{l\times m}$, the tensor rank $rank(A)$ is exactly the matrix rank and, for $A \in \mathbb{R}^{l\times m}$, the equality $rank_{\mathbb{R}}(A) = rank_{\mathbb{C}}(A)$ is valid. On the other hand, the tensor rank $rank(A)$, where $A \in \mathbb{F}^{l\times m\times n}$, depends on a field $\mathbb{F}$. It is clear that the inequality $rank_{\mathbb{C}}(A) \leqslant rank_{\mathbb{R}}(A)$ holds.

Example. Let

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{array}\right].$$

It can be shown that $rank_{\mathbb{R}}(A) = 3$ and $rank_{\mathbb{C}}(A) = 2$ (see [2, Example 3.44]).

Then, we introduce the topological group, which is the Cartesian product of general linear groups

$$GL_{l,m,n}(\mathbb{F}) := GL_l(\mathbb{F}) \times GL_m(\mathbb{F}) \times GL_n(\mathbb{F})$$

and consider a $GL_{l,m,n}(\mathbb{F})$-action on the space $\mathbb{F}^{l\times m\times n}$ (see also [4, Section 2.1]). For the notions and facts in the theory of topological transformation groups, we refer the reader, for example, to [12] and [13].

Let us take elements $A \in \mathbb{F}^{l\times m\times n}$ and $(L, M, N) \in GL_{l,m,n}(\mathbb{F})$ given as follows:

$$A = (a_{ijk}), \quad L = (\lambda_{pi}), \quad M = (\mu_{qj}), \quad N = (\nu_{rk}).$$

The tensor A is transformed into the tensor $B = (L, M, N)\cdot A \in \mathbb{F}^{l\times m\times n}$ by the rule:

$$B = (b_{pqr}) \in \mathbb{F}^{l\times m\times n}, \quad \text{where} \quad b_{pqr} = \sum_{i,j,k=1}^{l,m,n}\lambda_{pi}\mu_{qj}\nu_{rk}a_{ijk}.$$

Thus, we have the mapping called *the multilinear matrix multiplication*

$$\Phi\colon GL_{l,m,n}(\mathbb{F}) \times \mathbb{F}^{l\times m\times n} \longrightarrow \mathbb{F}^{l\times m\times n} : ((L, M, N), A) \longmapsto (L, M, N)\cdot A,$$

which was studied in [4, Sections 2.1, 2.2, and 2.5]).

The mapping Φ is considered below from the viewpoint of transformation groups.

Proposition 1. *The following properties are fulfilled:*

1) *the triple* $\langle GL_{l,m,n}(\mathbb{F}), \mathbb{F}^{l\times m\times n}, \Phi\rangle$ *is a topological transformation group;*
2) *every orbit for the* $GL_{l,m,n}(\mathbb{F})$*-action consists of elements of the same tensor rank;*
3) *the group action of* $GL_{l,m,n}(\mathbb{F})$ *on* $\mathbb{F}^{l\times m\times n}$ *is non-effective;*
4) *the space* $\mathbb{F}^{l\times m\times n}$ *is non-homogeneous under the* $GL_{l,m,n}(\mathbb{F})$*-action.*

Proof. 1) We show only the continuity of the multilinear matrix multiplication Φ. To this end, we take sequences $\{(L_t, M_t, N_t)\} \subset GL_{l,m,n}(\mathbb{F})$ and $\{A_t\} \subset \mathbb{F}^{l\times m\times n}$, $t \in \mathbb{N}$, that converge to $(L, M, N) \in GL_{l,m,n}(\mathbb{F})$ and $A \in \mathbb{F}^{l\times m\times n}$, respectively. Hence, we have the coordinatewise convergence:

$$\lim_{t\to+\infty} L_t = L, \quad \lim_{t\to+\infty} M_t = M, \quad \lim_{t\to+\infty} N_t = N.$$

We introduce the following notations:

$$L_t = (\lambda^t_{pi}), \quad M_t = (\mu^t_{qj}), \quad N_t = (\nu^t_{rk}), \quad A_t = (a^t_{ijk});$$

$$L = (\lambda_{pi}), \quad M = (\mu_{qj}), \quad N = (\nu_{rk}), \quad A = (a_{ijk}).$$

Then, we have the equalities for entries of tensors:

$$\lim_{t\to\infty} \lambda^t_{pi} = \lambda_{pi}, \quad \lim_{t\to\infty} \mu^t_{qj} = \mu_{qj}, \quad \lim_{t\to\infty} \nu^t_{rk} = \nu_{rk}, \quad \lim_{t\to\infty} a^t_{ijk} = a_{ijk}.$$

We define the constants as

$$M_1 = \sup_{p,i,t} \left|\lambda^t_{pi}\right|, \quad M_2 = \sup_{q,j,t} \left|\mu^t_{qj}\right|, \quad M_3 = \sup_{r,k,t} \left|\nu^t_{rk}\right|, \quad M_4 = \sup_{i,j,k,t} \left|a^t_{ijk}\right|.$$

In addition, let us set $(L_t, M_t, N_t) \cdot A_t = (b^t_{ijk})$ and $(L, M, N) \cdot A = (b_{ijk})$.

Then, for every $\varepsilon > 0$ there exists $T \in \mathbb{N}$ such that for all $t > T$ we have the following inequalities:

$$\begin{aligned}
\left|b_{pqr} - b^t_{pqr}\right| &\le \left|\sum_{i,j,k=1}^{l,m,n} \lambda_{pi}\mu_{qj}\nu_{rk}a_{ijk} - \sum_{i,j,k=1}^{l,m,n} \lambda^t_{pi}\mu^t_{qj}\nu^t_{rk}a^t_{ijk}\right| \le \\
&\le \sum_{i,j,k=1}^{l,m,n} \left|\lambda_{pi}\mu_{qj}\nu_{rk}a_{ijk} - \lambda^t_{pi}\mu^t_{qj}\nu^t_{rk}a^t_{ijk}\right| \le \\
&\le \sum_{i,j,k=1}^{l,m,n} \left(\left|\lambda_{pi} - \lambda^t_{pi}\right| \left|\mu_{qj}\nu_{rk}a_{ijk}\right| + \left|\mu_{qj} - \mu^t_{qj}\right| \left|\lambda^t_{pi}\nu_{rk}a_{ijk}\right|\right) + \\
&+ \sum_{i,j,k=1}^{l,m,n} \left(\left|\nu_{rk} - \nu^t_{rk}\right| \left|\lambda^t_{pi}\mu^t_{qj}a_{ijk}\right| + \left|a_{ijk} - a^t_{ijk}\right| \left|\lambda^t_{pi}\mu^t_{qj}\nu^t_{rk}\right|\right) \le \\
&\le \sum_{i,j,k=1}^{l,m,n} \left(\frac{\varepsilon M_2M_3M_4}{4lmnM_2M_3M_4} + \frac{\varepsilon M_1M_3M_4}{4lmnM_1M_3M_4} + \frac{\varepsilon M_1M_2M_4}{4lmnM_1M_2M_4} + \frac{\varepsilon M_1M_2M_3}{4lmnM_1M_2M_3}\right) \le \\
&\le \sum_{i,j,k=1}^{l,m,n} \frac{\varepsilon}{lmn} \le \varepsilon.
\end{aligned}$$

Hence, the sequence $\Phi((L_t, M_t, N_t), A_t)$ converges to the tensor $\Phi((L, M, N), A)$, as required.

2) See the proof of Lemma 2.3(2) in [4].

3) Take $(L, M, N) = (\alpha E_l, \beta E_m, \gamma E_n)$ with arbitrary scalars $\alpha, \beta, \gamma \in \mathbb{F}$ satisfying the condition $\alpha\beta\gamma = 1$. Then, for every $A = (a_{ijk}) \in \mathbb{F}^{l\times m\times n}$, we have

$$\Phi((L, M, N), A) = (\alpha\beta\gamma a_{ijk}) = (a_{ijk}) = \Phi((E_l, E_m, E_n), A).$$

This shows that the action is non-effective.

4) Consider tensors $A = \mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{z}_1$ and $B = \mathbf{x}_1 \otimes \mathbf{y}_1 \otimes \mathbf{z}_1 + \mathbf{x}_2 \otimes \mathbf{y}_2 \otimes \mathbf{z}_2$, where $\{\mathbf{x}_1, \mathbf{x}_2\} \subset \mathbb{F}^l, \{\mathbf{y}_1, \mathbf{y}_2\} \subset \mathbb{F}^m$ and $\{\mathbf{z}_1, \mathbf{z}_2\} \subset \mathbb{F}^n$ are pairs of linear independent vectors. Then, one has $rank\,(A) = 1$ and $rank\,(B) = 2$. Using item 2), we obtain the desired conclusion.

□

We have the following proposition on the semicontinuity of the tensor rank (see [4, Proposition 4.3, Theorem 4.10]).

Proposition 2. *The following properties are fulfilled:*
1) *for every $r \le \min(l,m)$, the set $\mathcal{S}_r(l,m) := \{A \in \mathbb{F}^{l\times m} | rank(A) \le r\}$ is closed;*
2) *there exists r such that the set $\mathcal{S}_r(l,m,n) := \{B \in \mathbb{F}^{l\times m\times n} | rank(B) \le r\}$ is not closed.*

2. The topological group of invertible matrices and approximations of matrices and tensors

In this section, we consider a low-rank approximation of tensors in the space $\mathbb{C}^{n\times n\times 2}$.

To this end, we should first formulate Bi's criterion for square-type tensors in the space $\mathbb{C}^{m\times m\times n}$ (see [14, Proposition 2.5]).

Proposition 3. *Let $A = [A_1|\dots|A_n]$ be a tensor in $\mathbb{C}^{m\times m\times n}$, where $n \geqslant 2$, and let $A_1 \in \mathbb{C}^{m\times m}$ be a nonsingular matrix. Then, the tensor rank of A is equal to m if and only if the matrices $A_2A_1^{-1}, \dots, A_nA_1^{-1}$ can be diagonalized simultaneously.*

Secondly, we recall some algebraic and topological definitions and facts about square matrices.

A matrix eigenvalue is said to be *simple* if its algebraic multiplicity equals one. *The spectrum* of a matrix is said to be *simple* provided that all eigenvalues of the given matrix are simple. In other words, if all eigenvalues are pairwise distinct. Certainly, the matrix with a simple spectrum is diagonalizable.

We recall that a mapping $f : X \to Y$ between two topological spaces is said to be *open* if for any open set O in X the image $f(O)$ is open in Y. For example, if a mapping $f : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is a surjective continuous homomorphism, then it is open [15, Theorem 5.29].

Using the topological group structure of the general linear group $GL_n(\mathbb{C})$, one can prove the following statement [9, Proposition 4].

Proposition 4. *Let $f_1, f_2, \dots, f_k : GL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$ be a finite family of self-mappings of the general linear group. Let us assume that at least one of these mappings is open with respect to the Euclidean topology. Let $A_1, A_2, \dots, A_k$ be arbitrary matrices in $M_n(\mathbb{C})$ and let $\|\cdot\|$ be a norm on $M_n(\mathbb{C})$. Then, for every $\varepsilon > 0$ there exists a finite family $A_{1\varepsilon}, A_{2\varepsilon}, \dots, A_{k\varepsilon}$ consisting of invertible matrices with simple spectra such that the inequalities*

$$\|A_1 - A_{1\varepsilon}\| < \varepsilon, \quad \|A_2 - A_{2\varepsilon}\| < \varepsilon, \dots, \quad \|A_k - A_{k\varepsilon}\| < \varepsilon$$

hold and the product matrix

$$f_1(A_{1\varepsilon})f_2(A_{2\varepsilon})\cdots f_k(A_{k\varepsilon})$$

has a simple spectrum.

For the case of two mappings, we put f_1 and f_2 to be the identity mapping and the inverse mapping, respectively:

$$f_1 : GL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C}) : X \longmapsto X; \quad f_2 : GL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C}) : X \longmapsto X^{-1}.$$

Obviously, both of these mappings are open with respect to the Euclidean topology in $GL_n(\mathbb{C})$. Therefore, as a consequence of the preceding proposition, we have

Corollary 1. *Let A and B be matrices $M_n(\mathbb{C})$ and let $\|\cdot\|$ be a norm on $M_n(\mathbb{C})$. Then, for every $\varepsilon > 0$ there exists a pair of matrices A_ε and B_ε in $GL_n(\mathbb{C})$ with simple spectra such that the inequalities*

$$\|A - A_\varepsilon\| < \varepsilon \quad \textit{and} \quad \|B - B_\varepsilon\| < \varepsilon$$

hold and the product matrix $A_\varepsilon B_\varepsilon^{-1}$ has a simple spectrum.

It is worth noting that one can use Corollary 1 for estimating the tensor rank of inverse matrices in the case when the given matrices are the factors of the Kronecker products (see [8]).

We make use of the above-mentioned results to prove the following assertion.

Proposition 5. Let A be a tensor in $\mathbb{C}^{n\times n\times 2}$ and let $\|\cdot\|$ be a norm on $\mathbb{C}^{n\times n\times 2}$. Then, the equality

$$\inf\left\{\|A - B\| : B \in \mathbb{C}^{n\times n\times 2} \quad \text{and} \quad \operatorname{rank}(B) = n\right\} = 0$$

holds, i.e., the tensor A may be approximated by tensors with tensor ranks equal to n.

Proof. We set $A = [A_1|A_2]$, where A_1 and A_2 are square matrices of size $n \times n$.

Let us fix $\varepsilon > 0$. Using Corollary 1, we take two invertible matrices B_1 and B_2 of size $n \times n$ such that the inequalities

$$\|A_1 - B_1\|_1 < \frac{\varepsilon}{2} \quad \text{and} \quad \|A_2 - B_2\|_1 < \frac{\varepsilon}{2}$$

hold and the product matrix $B_2 B_1^{-1}$ has a simple spectrum.

Let us consider the tensor $B = [B_1|B_2]$. By Bi's criterion, since the matrix $B_2 B_1^{-1}$ is diagonalizable, the tensor rank of B is equal to n. Moreover, we have the following estimation:

$$\|A - B\|_1 = \|A_1 - B_1\|_1 + \|A_2 - B_2\|_1 < \varepsilon.$$

In view of the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_1$, the rest is clear.

□

It is known (see [16], [17, Theorem 4.3]) that the maximum value of the tensor rank on the space $\mathbb{C}^{n\times n\times 2}$ is given by

$$mrank\,(n, n, 2) := \max\left\{rank\,(A) \,|\, A \in \mathbb{C}^{n\times n\times 2}\right\} = n + \left\lfloor \frac{n}{2} \right\rfloor,$$

where the symbol $\lfloor\cdot\rfloor$ means the integer part of a real number. Therefore, $mrank\,(2n, 2n, 2) = 3n$ for every $n \in \mathbb{N}$. This fact together with the Proposition 5 guarantees that, generally speaking, the tensor rank can leap an arbitrary large gap (see also [4, Section 4.5]). More precisely, we have

Corollary 2. *Let $n \in \mathbb{N}$. There exists a tensor $A \in \mathbb{C}^{2n\times 2n\times 2}$ with $rank\,(A) = 3n$ and a sequence of tensors $\{A_k\} \subset \mathbb{C}^{2n\times 2n\times 2}$, $k \in \mathbb{N}$, such that $rank\,(A_k) = 2n$ for every $k \in \mathbb{N}$ and*

$$\lim_{k\to+\infty} A_k = A,$$

where the limit is taken in the Euclidean topology.

Finally, we can conclude that the tensor rank is not semicontinuous on the tensor space $\mathbb{C}^{n\times n\times 2}$ endowed with the Euclidean topology.

Corollary 3. *Let* $n \geqslant 2$. *In the tensor space* $\mathbb{C}^{n\times n\times 2}$ *endowed with the Euclidean topology the set of tensors*

$$\left\{ T \in \mathbb{C}^{n\times n\times 2} \,|\, rank\,(T) \leqslant n \right\}.$$

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Аппроксимация тензорами малого ранга и топологическая группа обратимых матриц

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Аннотация

Под тензором понимается элемент тензорного произведения векторных пространств над некоторым полем. С точностью до выбора базисов в множителях тензорных произведений каждый тензор может быть снабжен координатами, то есть представлен в виде массива, состоящего из чисел. Статья посвящена свойствам тензорного ранга, который является естественным обобщением понятия матричного ранга. Существенную роль в изучении играет топологическая группа обратимых матриц. Полилинейное матричное умножение обсуждается с точки зрения групп преобразований. Рассматривается вопрос об аппроксимации тензорами малого ранга в конечномерных тензорных произведениях. Показывается, что задача о наилучшем приближении тензорами ранга n не имеет решения в пространстве тензоров размера $n\times n\times 2$. С этой целью используется аппроксимацию матрицами с простыми спектрами.

Ключевые слова: аппроксимация матрицами с простыми спектрами, аппроксимация тензорами малого ранга, действие группы, норма на тензорном пространстве, открытое отображение, простой спектр матрицы, топологическая группа обратимых матриц, топологическая группа преобразований, тензорный ранг

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