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ON EQUIVALENCE AND INVARIANTS OF PARTIAL LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN TWO VARIABLES UNDER CHANGE OF VARIABLES

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In the problem on a reduction of differential equations to a canonical form by the change of unknowns and variables, (relative) invariants with respect to these transformations arise in an intrinsic way (see [1], [2]). Invariants are known to be also important in the problem of a rough structure classification of differential equations (e. g., elliptic and hyperbolic quasilinear differential equations) (see [3]). Though in consideration of problems concerning linear partial differential equations some concrete invariants of various subclasses of these equations were used (for instance, the Laplace invariant) (see [4], [5]), nevertheless, invariants of the whole class of linear partial differential equations of second order are still unknown.

As for the ordinary linear differential equations of n-th order, the (relative) invariants were found with the use of various methods (see [1], [2], [6]–[11]). In particular, in [10] and [11] an algebraic method was suggested in order to describe the invariants of these equations over an ordinary differential field.

In the present article we consider linear partial differential equations of second order given over a differential field $(F, \partial_1, \partial_2)$. We introduce a natural analog of "variable change" (a change of differential operators ∂_1, ∂_2 by a special law), and reduce the equivalence problem for these equations to a simpler equivalence problem (see Theorem 1). By this theorem we describe the field of invariant differential rational functions of such equations over the field of constants of the field $(F, \partial_1, \partial_2)$ (Theorem 2). For the sake of simplicity all the differential fields under consideration are assumed to contain the field of rational numbers.

For the general notions and results in differential algebra the reader should refer to [12], [13].

Under a change of variables $\xi = \xi(x, y)$, $\eta = (x, y)$ the equation

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + a(x,y)u_x + b(x,y)u_y = 0$$

is transformed into the equation of the same form

$$A_1(\xi,\eta)v_{\xi\xi} + B_1(\xi,\eta)v_{\xi\eta} + C_1(\xi,\eta)v_{\xi\eta} + a_1(\xi,\eta)v_\xi + b_1(\xi,\eta)v_\eta = 0,$$

and the operators of partial derivatives with respect to ξ, η and with respect to x, y are related via $\delta = g^{-1}\partial$, where ∂ (δ) is a column vector with the "coordinates" $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$ ($\delta_1 = \partial/\partial\xi$, $\delta_2 = \partial/\partial\eta$, respectively) and g is a matrix with the entries $g_{11} = \xi_x$, $g_{12} = \eta_x$, $g_{21} = \xi_y$, $g_{22} = \eta_y$.

In the case of an arbitrary differential field $(F, \partial_1, \partial_2)$, where $\partial_1\partial_2 = \partial_2\partial_1$, the same thing can be represented as follows. Let

$$A\partial_1^2 u + B\partial_1\partial_2 u + C\partial_2^2 u + a\partial_1 u + b\partial_2 u = 0 \tag{1}$$