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A REAL ANALOG OF BRYANT'S TRANSFORMATION AND RATIONAL INTEGRAL CURVES OF A GIVEN DISTRIBUTION IN $\mathbb{P}^3$

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1. Introduction. We study the contact distribution on $\mathbb{P}^3$ comparing the geometry of this distribution with the geometry of an embedded into $\mathbb{P}^3$ integral manifold.

The most interesting three-dimensional real homogeneous contact manifolds are: 1) $(\mathbb{P}(T^*\mathbb{P}^2),\theta)$, where $\theta = x^0dy_0 + x^1dy_1 + x^2dy_2$; 2) $(\mathbb{P}^3,\omega)$, where $\omega = x^3dx^0 - 3x^2dx^1 + 3x^1dx^2 - x^0dx^3$.

By the Darboux theorem ([1], p. 328), three-dimensional contact manifolds are locally isomorphic. Since $(\mathbb{P}^3,\omega)$ and $(\mathbb{P}(T^*\mathbb{P}^2),\theta)$ are examples of algebraic varieties, it is natural to put the question on existence of a birational isomorphism between them, i. e., a not everywhere defined isomorphism which preserves the contact structures and is given by rational functions. A simple birational isomorphism which preserves the contact structures was found by R. Bryant in [2]. In [2], the problem on conformal representation of two-dimensional minimal surfaces was studied and it was proved that all such surfaces can be obtained from Riemannian surfaces. As important tools, R. Bryant used a birational isomorphism f from $\mathbb{P}(T^*\mathbb{CP}^2)$ to $\mathbb{CP}^3$ with standard contact structures and the following

Theorem 1 ([2]). *Let C be a contact curve in $\mathbb{CP}^3$. Then either C is a straight line or it is of the form $f(\widetilde{D})$, where $\widetilde{D} \subset \mathbb{P}(T^*\mathbb{CP}^2)$ is the horizontal lift of a reduced and irreducible plane curve $D \subset \mathbb{CP}^2$ of degree at least* 2.

With the use of the isomorphism f, each complex rational integral curve was obtained from a rational algebraic curve on the complex plane by two operations: 1) the lift of a curve to $\mathbb{P}(T^*\mathbb{P}^2)$; 2) the birational map of this manifold to $\mathbb{P}^3$.

R. Bryant's method gives a possibility to solve various problems. One of such problems is that of constructing explicit examples of homogeneous smooth Legendrian varieties, which can be easily solved [3] with the use of R. Bryant's construction.

The main goal of this paper is to transfer R. Bryant's construction into the real domain and to give its geometric interpretation. A real analog of R. Bryant's transformation is found in Item 2.

Starting with the papers of A. Voss, D.N. Sintsov [4], integral curves of the given distribution play an important role in the study of contact structures on three-dimensional manifolds. Since the contact structure in $\mathbb{P}^3$ is determined by the normal curve, it is natural to characterize integral curves and their partial classes in terms of the normal curve. We solve this problem in Section 3.

The birational transformation suggested by R. Bryant connects the geometries of the manifolds and reduces, in a number of cases, problems concerning integral curves to the study of curves on the projective plane. This fact allows us to solve the problem on possibility of connecting two points of the three-dimensional projective space by a rational integral curve. The question on possibility of connecting two points by a piecewise rational integral curve has been set and solved by Chow (1939) and Rashevskii (1938). In 1963, Smale raised the question on possibility of connecting two points by a C^∞-smooth curve. This problem has been solved by V.N. Chernenko [5]. This problem