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On the A. M. Bikchentaev Conjecture

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Abstract—In 1998 A. M. Bikchentaev conjectured that for positive τ-measurable operators a and b affiliated with a semifinite von Neumann algebra, the operator $b^{1/2}ab^{1/2}$ is submajorized by the operator ab in the sense of Hardy—Littlewood. We prove this conjecture in its full generality and obtain a number of consequences for operator ideals, Golden—Thompson inequalities, and singular traces.

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1. INTRODUCTION

In this paper we answer a question due to A. M. Bikchentaev (see [1, P. 573, conjecture A]) in the affirmative. To formulate his conjecture, we need some notions from the theory of noncommutative integration. For details on von Neumann algebra theory, the reader is referred to e.g. [2] or [3]. General facts concerning τ−measurable operators may be found in [4]. We recall basic definitions for convenience of the reader.

In what follows, H is a Hilbert space and $B(H)$ is the $*$-algebra of all bounded linear operators on H. Let $\mathcal{M}$ be a von Neumann algebra on H.

Let $\mathcal{M}$ be a semifinite von Neumann algebra, let τ be a faithful normal semi-finite trace on $\mathcal{M}$ and let $S(\mathcal{M},\tau)$ be a $*$-algebra of τ-measurable operators. For every τ-measurable operator x we define the function $\mu(x):\mathbb{R}_+\to\mathbb{R}_+$ as follows

$$\mu(s;x)=\inf\{\|xp\| : p\in\mathcal{M} \text{ is a projector and } \tau(1-p)\le s\}.$$

Let $\mathcal{M}=B(H)$ ($\mathcal{M}=\ell_\infty$, respectively) and let Tr be the standard trace (counting measure, respectively). It is easy to show that $S(\mathcal{M},\tau)=\mathcal{M}$. For every $x\in S(\mathcal{M},\tau)$ we have

$$\mu(n;x)=\mu(t;x),\quad t\in[n,n+1),\quad n\ge 0.$$

The numbers $\mu(n;x)$ and $n\ge 0$ are eigenvalues of the operator $|x|=(x^*x)^{1/2}$.

Let a and b be τ−measurable operators. The element a majorizes the element b in the sense of Hardy—Littlewood—Polya (write $b\prec\prec a$), if

$$\int_0^t \mu(s;b)ds \le \int_0^t \mu(s;a)ds \ \ \forall t>0.$$

Evidently, $b\prec\prec a$ if and only if $\mu(b)\prec\prec\mu(a)$.

In the special case when a and b are positive operators the following question was asked by A. M. Bikchentaev.

Question. *Let $a,b\ge 0$ be self-adjoint operators from $S(\mathcal{M},\tau)$. Is it necessarily true that*

$$b^{1/2}ab^{1/2}\prec\prec ab\,?$$

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