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LIFTING OF RIEMANNIAN METRICS TO TENSOR BUNDLES

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1. Introduction

In this paper, we define the diagonal lift ${}^Dg$ of a Riemannian metric g from a smooth manifold M_n to the tensor bundle $T^0_q(M_n)$ and study Killing vector fields and geodesics of the metric ${}^Dg$.

Let M_n be an n-dimensional differentiable manifold of class C^∞ and $T^0_q(M_n)$ the tensor bundle of type $(0,q)$ over M_n. In terms of local coordinates on M_n defined in a neighborhood $U \subset M_n$, a tensor t of type $(0,q)$ at $x \in U$ is determined by the collection of numbers $(x^i, t_{i_1\dots i_q})$, where x^i are the coordinates of x and $t_{i_1\dots i_q}$ the components of t with respect to the natural frame at x. The collections $(x^i, t_{i_1\dots i_q}) = (x^i, x^{\overline{i}}) = (x^I)$, $i = 1,\dots,n$, $\overline{i} = n+1,\dots,n+n^q$, $I = 1,\dots,n+n^q$, are the local coordinates of elements of the bundle $T^0_q(M_n)$ in the neighborhood $\pi^{-1}(U)$, where π is the natural projection of $T^0_q(M_n)$ to M_n.

Let now M_n be a Riemannian manifold with metric g, g_{ji} the components of g with respect to the local coordinates in U, and Γ^h_{ji} the Christoffel symbols of g in U. Let $F(M_n)$ denote the ring of differentiable functions of class C^∞ on M_n, and $\mathcal{T}^r_s(M_n)$ denote the vector space of all C^∞ differentiable tensor fields of type (r,s) on M_n. $\mathcal{T}^r_s(M_n)$ is a module over $F(M_n)$. Consider a vector field $X \in \mathcal{T}^1_0(M_n)$ and a tensor field $w \in \mathcal{T}^0_q(M_n)$. Let X^h and $w_{h_1\dots h_q}$ be the local components of X and w respectively. The complete lift ${}^CX \in \mathcal{T}^1_0(T^0_q(M_n))$ of X, the horizontal lift ${}^HX \in \mathcal{T}^1_0(T^0_q(M_n))$ of X and the vertical lift ${}^Vw \in \mathcal{T}^1_0(T^0_q(M_n))$ of w to the bundle $T^0_q(M_n)$ have, respectively, the following components with respect to the natural frame $\{\partial_H\} = \{\partial_h, \partial_{\overline{h}}\}$, $x^{\overline{h}} = t_{h_1\dots h_q}$, on $T^0_q(M_n)$ [1]–[3]:

$$
\begin{aligned}
{}^CX &= \begin{pmatrix} X^h \\ -\sum\limits_{\mu=1}^{q} t_{h_1\dots m\dots h_q}\partial_{h_\mu}X^m \end{pmatrix}, \quad {}^Vw = \begin{pmatrix} 0 \\ w_{h_1\dots h_q} \end{pmatrix}, \\
{}^HX &= \begin{pmatrix} X^h \\ X^m \sum\limits_{\mu=1}^{q} \Gamma^s_{mh_\mu} t_{h_1\dots s\dots h_q} \end{pmatrix}.
\end{aligned}
\tag{1}
$$

For each coordinate neighborhood $U(x^h)$ on M_n, we define

$$
X_j = \frac{\partial}{\partial x^j} = \delta^h_j \frac{\partial}{\partial x^h} \in \mathcal{T}^1_0(M_n), \quad j = 1,\dots,n,
$$
$$
w_{\overline{j}} = dx^{j_1} \otimes \dots \otimes dx^{j_q} = \delta^{j_1}_{h_1} \dots \delta^{j_q}_{h_q} dx^{h_1} \otimes \dots \otimes dx^{h_q} \in \mathcal{T}^0_q(M_n), \quad \overline{h} = n+1,\dots,n+n^q.
$$