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ON SOLVABILITY CONDITIONS FOR OVERDETERMINED SYSTEMS OF DIFFERENTIAL EQUATIONS IN SOBOLEV SPACES

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Introduction

Let X be a smooth n-dimensional manifold, E_i $(0 \le i \le N < \infty)$ vector bundles over X, and $\{A_i, E_i\}$ an elliptic complex of (linear) differential operators with smooth coefficients on X:

$$0 \to C^\infty(E_0) \xrightarrow{A_0} C^\infty(E_1) \xrightarrow{A_1} \cdots \xrightarrow{A_N} C^\infty(E_{N+1}) \to 0. \tag{1}$$

This means that $A_{i+1} \circ A_i = 0$ and the sequence

$$0 \to (E_0)_x \xrightarrow{\sigma(A_0)(x,\zeta)} (E_1)_x \xrightarrow{\sigma(A_1)(x,\zeta)} \cdots \xrightarrow{\sigma(A_N)(x,\zeta)} (E_{N+1})_x \to 0$$

is exact for all $x \in X$ and all $\zeta \in T^*(X) \setminus \{0\}$, where $\sigma(A_i)(x, \zeta)$ is the principal symbol of the operator A_i and $T^*(X)$ is a real cotangent bundle over X. For years the problem of local acyclicity of such complexes is one of basic unsolved problems of the theory of overdetermined systems (see, e. g., [1]).

In this paper, for complexes consisting of operators of the same order $m \ge 1$, we study solvability conditions for the equation $A_i u = f$ in Sobolev spaces on open subsets of X. For this purpose, we suggest to use iterations of Green's integrals (cf. [2] and [3] for elliptic systems). This method allows us not only to obtain solvability conditions, but also to find formulas for H^m-solutions of the equation $A_i u = f$ if such solutions exist. The solutions are given in the form of the sum of a series whose addends are iterations of pseudodifferential operators constructed with the use of special fundamental solutions of the Laplacians of the complex. For the Dolbeault complex, this operators are similar to the Martinelli–Bochner integral [4].

1. Green's integrals for elliptic complexes

Denote by X a smooth compact manifold with (possibly empty) smooth boundary ∂X embedded into a smooth manifold $\widetilde{X}$ of the same dimension, and by $\mathring{X}$ the interior of X. Fixing $i \ge 0$, we will study the complex (1) in degree i.

Denote by $A^*_{i-1} \in \mathrm{Diff}_{m_{i-1}}(E_i \to E_{i-1})$ the operator formally adjoint to A_{i-1} with respect to Hermitian metrics $(\cdot,\cdot)_{x,i}$, $(\cdot,\cdot)_{x,i-1}$ in the fibers of E_i and E_{i-1} respectively, where $\mathrm{Diff}_m(E \to F)$ is the set of all differential operators of order not greater than m between the bundles E and F.

Here and in what follows we will assume that the operators A_i and A_{i-1} are of the same order $m \ge 1$. Although this assumption restricts the domain of application of the results, note that, for any elliptic complex, there exists a homotopically equivalent complex consisting only of first order operators ([1], p. 34–40). Since the complex $\{A_i, E_i\}$ is elliptic, on this assumption, the (principal)

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