ISSN 1066-369X, Russian Mathematics, 2018, Vol. 62, No. 5, pp. 10–15. © Allerton Press, Inc., 2018.
Original Russian Text © A.M. Bikchentaev, S.A. Abed, 2018, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2018, No. 5, pp. 13–19.

Paranormal Elements in Normed Algebra

A. M. Bikchentaev* **and S. A. Abed****

Kazan Federal University
ul. Kremlyovskaya 18, Kazan, 420008 Russia

Received March 29, 2017

Abstract—For a normed algebra $\mathcal{A}$ and natural numbers k we introduce and investigate the $\|\cdot\|$-closed classes $\mathcal{P}_k(\mathcal{A})$. We show that $\mathcal{P}_1(\mathcal{A})$ is a subset of $\mathcal{P}_k(\mathcal{A})$ for all k. If T in $\mathcal{P}_1(\mathcal{A})$, then T^n lies in $\mathcal{P}_1(\mathcal{A})$ for all natural n. If $\mathcal{A}$ is unital, $U, V \in \mathcal{A}$ are such that $\|U\| = \|V\| = 1$, $VU = I$ and T lies in $\mathcal{P}_k(\mathcal{A})$, then UTV lies in $\mathcal{P}_k(\mathcal{A})$ for all natural k. Let $\mathcal{A}$ be unital, then 1) if an element T in $\mathcal{P}_1(\mathcal{A})$ is right invertible, then any right inverse element T^{-1} lies in $\mathcal{P}_1(\mathcal{A})$; 2) for $\|I\| = 1$ the class $\mathcal{P}_1(\mathcal{A})$ consists of normaloid elements; 3) if the spectrum of an element T, $T \in \mathcal{P}_1(\mathcal{A})$ lies on the unit circle, then $\|TX\| = \|X\|$ for all $X \in \mathcal{A}$. If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then the class $\mathcal{P}_1(\mathcal{A})$ coincides with the set of all paranormal operators on a Hilbert space $\mathcal{H}$.

DOI: 10.3103/S1066369X1805002X

Keywords: *Hilbert space, C^*-algebra, paranormal operator, quasinilpotent operator, isometry, hyponormal operator, normaloid operator, normed algebra, unital algebra.*

Introduction. Investigation of different subsets in normed algebras and in $*$-algebras of operators is an actual problem of functional analysis (see, e.g., [1–5] for classes of hyponormal, normal, idempotent, unitary operators and differences of idempotents, respectively). In this paper for a normed algebra $\mathcal{A}$ and $k \in \mathbb{N}$ we introduce and investigate $\|\cdot\|$-closed classes

$$\mathcal{P}_k(\mathcal{A}) = \{T \in \mathcal{A} : \|T^{k+1}A\| \geq \|TA\|^{k+1} \text{ for all } A \in \mathcal{A} \text{ with } \|A\| = 1\}.$$

It is shown that $\mathcal{P}_1(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{A})$ for all $k \in \mathbb{N}$ (Theorem 2). If $\mathcal{A}$ is a dense subalgebra of normed algebra $\mathcal{B}$, then $\mathcal{P}_k(\mathcal{A}) \subset \mathcal{P}_k(\mathcal{B})$ for all $k \in \mathbb{N}$ (Proposition 1). If $T \in \mathcal{P}_1(\mathcal{A})$, then $T^n \in \mathcal{P}_1(\mathcal{A})$ for all $n \in \mathbb{N}$ (Theorem 5). If $\mathcal{A}$ is unital, $U, V \in \mathcal{A}$ are such that $\|U\| = \|V\| = 1$, $VU = I$ and $T \in \mathcal{P}_k(\mathcal{A})$, then $UTV \in \mathcal{P}_k(\mathcal{A})$ for all $k \in \mathbb{N}$ (Theorem 3). In particular, if $\mathcal{A}$ is a unital C^*-algebra and $T \in \mathcal{P}_k(\mathcal{A})$, then $UTU^* \in \mathcal{P}_k(\mathcal{A})$ for all isometries $U \in \mathcal{A}$ and $k \in \mathbb{N}$ (Corollary 3). If $\mathcal{A}$ is commutative and $\|T^2\| = \|T\|^2$ for all $T \in \mathcal{A}$, then $\mathcal{P}_1(\mathcal{A}) = \mathcal{A}$ (Proposition 6).

Let $\mathcal{A}$ be unital, then 1) if an element $T \in \mathcal{P}_1(\mathcal{A})$ is right invertible, then any right inverse element T^{-1} lies in $\mathcal{P}_1(\mathcal{A})$ (Theorem 4); 2) for $\|I\| = 1$ the class $\mathcal{P}_1(\mathcal{A})$ consists of normaloid elements (Corollary 1); 3) if the spectrum of an element T, $T \in \mathcal{P}_1(\mathcal{A})$ lies on the unit circle, then $\|TX\| = \|X\|$ for all $X \in \mathcal{A}$ (Corollary 4). If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then the class $\mathcal{P}_1(\mathcal{A})$ coincides with the set of all paranormal operators on a Hilbert space $\mathcal{H}$ (Corollary 6).

1. Notations and definitions. An *algebra* is a vector space $\mathcal{A}$ over the field Λ ($= \mathbb{R}$ or $\mathbb{C}$), equipped with a bilinear product such that

$$X(YZ) = (XY)Z, \quad (Y + Z)X = YX + ZX,$$

$$X(Y + Z) = XY + XZ, \quad \lambda(XY) = (\lambda X)Y = X(\lambda Y)$$

for all $X, Y, Z \in \mathcal{A}$ and $\lambda \in \Lambda$. An algebra $\mathcal{A}$ is *unital* (i.e., possesses the unity), if there exists an element $(0 \neq) I \in \mathcal{A}$ such that $IX = XI = X$ $(X \in \mathcal{A})$. An element X of algebra $\mathcal{A}$ with I is said to be *right invertible*, if there exists an element $X^{-1} \in \mathcal{A}$ such that $XX^{-1} = I$. An algebra $\mathcal{A}$ is said to

*E-mail: Airat.Bikchentaev@kpfu.ru.
**E-mail: samialbarkish@gmail.com.