Проявления теоремы Бэра-Капланского в теории абелевых групп без кручения

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Baer-Kaplansky theorem:
Let $X, Y \in C$. Then $X \cong Y$ if and only if $\mathrm{End}(X) \cong \mathrm{End}(Y)$.

Strong Baer-Kaplansky theorem:
Let $X, Y \in C$. Then $X \cong Y$ if and only if $\mathrm{End}(X) \cong \mathrm{End}(Y)$, moreover, any ring isomorphism $\alpha : \mathrm{End}(X) \longrightarrow \mathrm{End}(Y)$ is induced by a group isomorphism $\phi : X \longrightarrow Y$ in the following way: $F\alpha = \phi^{-1}F\phi$ for any $F \in \mathrm{End}(X)$.

Weak Baer-Kaplansky theorem:
Let $X, Y \in C$. Then $X \cong_{nr} Y$ if and only if $\mathrm{End}(X) \cong_{nr} \mathrm{End}(Y)$.

$$CD_n \subset CRQ_n \subset ACD_n \subset TFAG_n$$

$X \in TFAG_n$ if $\mathbb{Z}^n \subset X \subset \mathbb{Q}^n = \mathbb{Q} \oplus \mathbb{Q} \ldots \oplus \mathbb{Q}$
(X is a torsion-free abelian group of rank n)

$A \in CD_n$ if $A = A_1 \oplus A_2 \ldots \oplus A_n$ with $A_i \in \mathbb{Q}$
(A is a completely decomposable abelian group of rank n)

$X \in ACD_n$ if there exists $A \in CD_n$ such that $A \subset X$ and $|X/A| < \infty$
(X is an almost completely decomposable abelian group of rank n)

$X \in CRQ_n$ if $X \in ACD_n$ and $|X/A|$ is a finite cyclic group
(X is an almost completely decomposable abelian group of rank n with cyclic regulator qoutient)

Example. Let $\tau_p = \langle \frac{1}{p}, \frac{1}{p^2}, \dots, \frac{1}{p^k}, \dots \rangle \subset \mathbb{Q}$ for any prime p and $\tau_{p_1} \oplus \tau_{p_2} \subset \mathbb{Q}^2$ be a completely decomposable (CD) group with $p_1 \neq p_2$.
Let $X = \langle \tau_{p_1} \oplus \tau_{p_2}, b \rangle \subset \mathbb{Q}^2$ with $b = \frac{(m,k)}{q}$ and prime q different from p_1 and p_2, also $\gcd(mk, q) = 1$.

CRQ-group X is indecomposable and $\mathrm{End}\,(X) \cong \langle I, q(\tau_{p_1} \oplus \tau_{p_2}) \rangle$ is a E-ring. Furthermore, $\mathrm{End}\,(X)^+ \cong (X)^+$ if $m = k = 1$, otherwise $\mathrm{End}\,(X)^+ \cong_{nr} (X)^+$.

Let $Y = \langle \tau_{p_1} \oplus \tau_{p_2}, c \rangle \subset \mathbb{Q}^2$ with $c = \frac{(m', k')}{h}$ and prime h different from q, p_1 and p_2. Let $G = X \oplus Y \subset \mathbb{Q}^4$.
Regulator: $R(G) \cong \tau_{p_1} \oplus \tau_{p_1} \oplus \tau_{p_2} \oplus \tau_{p_2}$,
Regulator quotient: $G/R(G) \cong \mathbb{Z}/q\mathbb{Z} \oplus \mathbb{Z}/h\mathbb{Z}$.
$G \cong_{nr} \langle \tau_{p_1} \oplus \tau_{p_2}, b, c \rangle \oplus \tau_{p_1} \oplus \tau_{p_2}$:

$$4 = 2 + 2 = 2 + 1 + 1$$

Definition Let G and H be torsion-free abelian groups of finite rank. Then G and H are called nearly isomorphic (in symbols $G \cong_{nr} H$) if and only if for any prime q there are monomorphisms $\eta_q : G \longrightarrow H$ and $\xi_q : H \longrightarrow G$ such that $H/\eta_q(G)$ and $G/\xi_q(H)$ are finite groups and $|H/\eta_q(G)|$ and q as well as $|G/\xi_q(H)|$ and q are relatively prime.

Arnold (1982): If $G \cong_{nr} H$ and $G = G_1 \oplus \ldots \oplus G_k$ then there exists a decomposition $H = H_1 \oplus \ldots \oplus H_k$ such that $G_1 \cong_{nr} H_1$, ... , $G_k \cong_{nr} H_k$.

B. Jonsson (1957,1959), A.L.S. Corner (1961, 1969)

$X = X_1 \oplus \ldots \oplus X_s = Y_1 \oplus \ldots \oplus Y_t, \quad s \neq t.$
$\mathrm{rk}\,(X) = \mathrm{rk}\,(X_1) + \ldots + \mathrm{rk}\,(X_s) = \mathrm{rk}\,(Y_1) + \ldots + \mathrm{rk}\,(Y_t)$

L. Fuchs, Problem 68: What partitions of a natural number $n = m_1 + m_2 + \ldots + m_s = l_1 + l_2 + \ldots + l_t$ can be associated with two different direct decompositions of a torsion-free abelian group X of rank n into indecomposable summands of ranks $m_1, m_2, \ldots, m_s$ and $l_1, l_2, \ldots, l_t$?

E. Blagoveshchenskaya, A. Yakovlev (1989):
$m_i \leq R(n,t)$ and $l_j \leq R'(n,s)$

E. Blagoveshchenskaya (1983):
$X \in CRQ$, $m_i \leq n - t + 1$ and $l_j \leq n - s + 1$

$X \in ACD, \quad R(X) = A = \bigoplus_{i=1,\dots,n} A_i$ with $A_i \subset \mathbb{Q}$,
$e(X/A) = 0, \quad e = \prod_{p \in P} p^{\gamma_p}$.

$eX \subset A$:

$$A \subset X \subset \frac{A}{e} \quad \text{with} \quad e\frac{A}{e} = A.$$

$$\frac{A}{e} = \sum_{p \in P} \frac{A}{p^{\gamma_p}} \quad \text{with} \quad \frac{A}{{p_1}^{\gamma_{p_1}}} \cap \frac{A}{{p_2}^{\gamma_{p_2}}} = A \quad \text{if} \quad p_1 \neq p_2$$

$X = \sum_{p \in P} X_p$ with fully invariant subgroups $X_p = X \cap \frac{A}{p^{\gamma_p}}$ of X and each X_p is fully invariant in X $(p \in P)$

Primary factor representations of X and $\mathrm{End}\,(X)$ accordingly:

$$X = \sum_{p \in P} X_p \quad \text{with} \quad A \subset X_p \subset \frac{A}{p^{\gamma_p}}$$

$X, X_p \in ACD, \quad A \subset X \subset \frac{A}{e}, \quad \mathrm{rk}\, X = \mathrm{rk}\, X_p = n,$
$A = R(X) = R(X_p), \quad p^{\gamma_p} X_p \subset A, \quad eX \subset A.$

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$$\mathrm{End}\,(X) = \bigcap_{p \in P} \mathrm{End}\,(X_p) \quad \text{with} \quad p^{\gamma_p}\mathrm{End}\,(A) \subset \mathrm{End}\,(X_p) \subset \mathrm{End}\,(A)$$

$$e\mathrm{End}\,(A) \subset \mathrm{End}\,(X) \subset \mathrm{End}\,(A) \quad \text{with} \quad e = \textstyle\prod_{p \in P} p^{\gamma_p}$$

$\mathrm{End}\,(A)^+ \in CD, \quad n < \mathrm{rk}\,(\mathrm{End}\,(A)^+) < n^2, \quad A \subset \mathrm{End}\,(A)^+,$
$\mathrm{End}\,(X_p)^+ \in ACD, \quad \mathrm{End}\,(X)^+ \in ACD$

Rigid ACD group X with primary regulator quotient:

Let us consider a group X of rank n ($n \geq 3$), which contains a completely decomposable group A, such that X/A is a primary p-group of rank $t < n$ and exponent $e = p^l$.
Let $A = A_{\tau_1} \oplus A_{\tau_2} \oplus \ldots \oplus A_{\tau_n}$ be the decomposition of A into τ_j-homogeneous components of rank 1, with $\{\tau_1, \ldots, \tau_n\}$ a set of pairwise incomparable idempotent types, which are not p-divisible.
Then $A \subset X \subset \frac{A}{p^l}$, $p^l \mathrm{End}\,(A) \subset \mathrm{End}\,(X) \subset \mathrm{End}\,(A)$ with

$$\mathrm{End}\,A = \Pi_{j=1,\ldots,n}\mathrm{End}\,(A_{\tau_j}) \cong \Pi_{j=1,\ldots,n}\mathrm{End}\,({\tau_j}^+) \cong \Pi_{j=1,\ldots,n}\tau_j.$$

Level structures of X and $\mathrm{End}\,(X)$:

Denote $X_k = \frac{A}{p^k} \cap X \quad (p^k X_k \subset A)$ and $\mathcal{E}_A{}^{(k)} = \mathrm{End}\,(X_k)$; $k = 0, 1, \ldots, l$:

$$X = X_l \supset X_{l-1} \supset X_{l-2} \supset X_{l-3} \supset \ldots \supset X_2 \supset X_1 \supset X_0 = A.$$

$$\mathrm{End}\,X = \mathcal{E}_A{}^{(l)} \subset \mathcal{E}_A{}^{(l-1)} \subset \mathcal{E}_A{}^{(l-2)} \subset \ldots \subset \mathcal{E}_A{}^{(1)} \subset \mathcal{E}_A{}^{(0)} = \mathrm{End}\,A$$

Automorphism invariance areas for any $\mathcal{B} \in \mathrm{Aut}\,(\mathcal{E}) \subset \mathrm{Aut}\,(\mathcal{E}_A)$:

$$\mathcal{E}_A = (\mathcal{E}_A{}^{(0)} \setminus \mathcal{E}_A{}^{(1)}) \cup (\mathcal{E}_A{}^{(1)} \setminus \mathcal{E}_A{}^{(2)}) \cup \ldots \cup (\mathcal{E}_A{}^{(l-1)} \setminus \mathcal{E}_A{}^{(l)}) \cup \mathcal{E}_A{}^{(l)}$$

Mader (1990-2000):

Let $X, Y \in ACD$ and $X \cong_{nr} Y$, then $R(X) = R(Y) = A$ and $X/A \cong Y/A$.

Let $e = \exp X/A = \exp Y/A$ then $eX \subset A$ and $eY \subset A$.

Denote $\overline{} : A \longmapsto A/eA = \overline{A}$.

Clearly, $\overline{eX} = eX/eA \cong eY/eA = \overline{eY}$.

$X \cong_{nr} Y$ if and only if there exists a type-automorphism $\phi : \overline{A} \longrightarrow \overline{A}$ such that $\overline{eX}\phi = \overline{eY}$.

E. Blagoveshchenskaya, A. Mader (1994):
Classsification of CRQ-groups up to near-isomorphism:
E. Blagoveshchenskaya, A. Mader (1994):

Let $X \in CRQ$: $\quad X = \langle A, u \rangle$ with $eu = \sum_{\tau \in T} v_\tau$, $v_\tau \in A_\tau$.
For any $\tau \in T$ define $m_\tau(X) = |\overline{v_\tau}| = |v_\tau + eA|$.

Theorem

Let $X, Y \in CRQ$.
$X \cong_{nr} Y$ if and only if $R(X) \cong R(Y)$ and $m_\tau(X) = m_\tau(Y)$ for any $\tau \in T$.

E. Blagoveshchenskaya (2004): Classification of CRQ-groups up to isomorphism.
E. Blagoveshchenskaya (2004):
Let $X, Y \in ACD$ and $X \cong_{nr} Y$. Then $\mathrm{End}\,(X)^+ \cong_{nr} \mathrm{End}\,(Y)^+$.

E. Blagoveshchenskaya, G. Ivanov, P. Schultz (2001):

Theorem in the near Baer-Kaplansky form

Let $X, Y \in CRQ$. Then $\mathrm{End}\,(X) \cong \mathrm{End}\,(Y)$ if and only if $X \cong_{nr} Y$.

Definition.

A rigid acd group X of ring type will be called a *strictly indecomposable group* if its envelope X' is indecomposable.

Theorem. Let X be a rigid acd group of ring type. Then there exists a rigid crq-group Y such that $\mathrm{End}\, X \cong \mathrm{End}\, Y$ if and only if X is strictly indecomposable.

X' is decomposable then X is not strictly indecomposable, but X is indecomposable in the usual sense:

$$\overline{p^2 X} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & p & p & 0 \end{pmatrix}, \tag{1}$$

$$\overline{p^2 X'} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{r} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

X is strictly indecomposable:

$$\overline{p^2 X} = \begin{pmatrix} 1 & 0 & p \\ 0 & p & -p \end{pmatrix}, \quad \overline{p^2 X'} = \begin{pmatrix} 1 & 0 & p \\ 0 & 1 & -1 \end{pmatrix}.$$

A torsion-free abelian group X belongs to the class of **Generalized CRQ-groups** if it contains a completely decomposable subgroup $R(X) = \bigoplus_{\tau \in T_{cr}(R(X))} A_\tau$ such that the following conditions are satisfied:

1. $T_{cr}(R(X))$ is a finite or countable set of pairwise incomparable types;
2. A_τ is a pure subgroup of X of finite rank for each $\tau \in T_{cr}(R(X))$;
3. $X/R(X) = \bigoplus_{p \in P_X} T_p^X$ for a finite or countable set of primes P_X and finite cyclic p-groups T_p^X;
4. for every $p \in P_X$ the set $\{q \in P_X : [T_p^X] \cap [T_q^X] \neq \emptyset\}$ is finite; here $[T_p^X]$ is the minimal subset $T_p \subset T_{cr}(R(X))$ satisfying $T_p^X \subseteq ((\bigoplus_{\tau \in T_p} A_\tau)_*^X + R(X))/R(X)$.

Near-isomorphism is expected

(1) to preserve decomposability properties of groups

(2) to be isomorphism for completely decomposable groups

(3) to be traditional near-isomorphism for groups of finite rank

Definition(Blagoveshchenskaya, Strüngmann, Göbel, 2002 – 2007)
Let G and H be torsion-free abelian groups. Then G and H are called **nearly isomorphic**, $G \cong_n H$ if for every integer N there exist monomorphisms $\varphi_N : G \to H$ and $\psi_N : H \to G$ such that

1. $H/G\varphi_N$ and $G/H\psi_N$ are torsion;
2. $(H/G\varphi_N)_p = 0 = (G/H\psi_N)_p$ for all primes p dividing N;
3. for every finite rank pure subgroups $G' \subseteq G$ and $H' \subseteq H$ the quotients $(G'\varphi_N)_*^H/G'\varphi_N$ and $(H'\psi_N)_*^G/H'\psi_N$ are finite.

E. Blagoveshchenskaya (2004):

Theorem

Let $X, Y \in ACD$ (or Generalized ACD) and $X \cong_{nr} Y$. Then $End(X)^{+} \cong_{nr} End(Y)^{+}$.

In other words, if $X, Y \in ACD$ then
$eEnd(A) \subset End(X) \subset End(A)$ and $eEnd(A) \subset End(Y) \subset End(A)$ with $eX \subset A$ and $eY \subset A$.
Let $\overline{} : End(A) \longmapsto End(A)/eEnd(A) = \overline{End(A)}$.
Clearly, $\overline{End(X)} \subset \overline{End(A)}$ *and* $\overline{End(Y)} \subset \overline{End(A)}$.

$End(X) \cong_{nr} End(Y)$ if and only if there exists a **ring** automorphism $\Phi : \overline{End(A)} \longrightarrow \overline{End(A)}$ such that $\overline{End(X)}\Phi = \overline{End(X)}$.

СПАСИБО