Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 45, No. 7, pp. 75–76, 2001 *UDC 517.948*

PHENOMENOLOGICAL SYMMETRY AND FUNCTIONAL EQUATIONS

G.G. Mikhaĭlichenko and V.M. Malyshev

Let a one-dimensional manifold $\mathfrak{M}$ and a function $f : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}$ which assigns to a pair $\langle i,j\rangle \in \mathfrak{M} \times \mathfrak{M}$ a number $f(i,j) \in \mathbb{R}$ be given. It is supposed that f is smooth and has non-vanishing derivatives. To any triple $\langle i,j,k\rangle \in \mathfrak{M}^3$ the function f assigns intrinsically the point $(f(i,j), f(i,k), f(j,k))$ of $\mathbb{R}^3$. It is said that f determines on $\mathfrak{M}$ a one-dimensional phenomenologically symmetric geometry (physical structure) of rank 3 if all such points lie in a two-dimensional hypersurface given by an equation $\Phi(f(i,j), f(i,k), f(j,k)) = 0$ with $\operatorname{grad}\Phi \neq 0$. Let us represent this equation in the form resolved with respect to the first argument

$$f(i,j) = \varphi(f(i,k), f(j,k)), \tag{1}$$

where $\varphi(u,v)$ is a function of two variables such that $\varphi_u \neq 0$, $\varphi_v \neq 0$.

Equation (1) fulfilled by any triple $\langle i,j,k\rangle$ is a mathematical expression of the principle of phenomenological symmetry, the basic postulate of the theory of physical structures suggested by Yu.I. Kulakov (see [1]) for analyzing the structure of physical laws.

Next, consider a quadruple $\langle i,j,k,l,\rangle$ and write down equation (1) for all triples formed by its elements: $\langle i,j,k\rangle$, $\langle i,j,l\rangle$, $\langle i,k,l\rangle$, $\langle j,k,l\rangle$. From these four equations one can easily derive the relation

$$\varphi[\varphi(f(i,l), f(k,l)), \varphi(f(j,l), f(k,l))] = \varphi(f(i,l), f(j,l)),$$

in which, evidently, the variables $f(i,l) = x$, $f(j,l) = y$, $f(k,l) = z$ are independent. Hence we obtain the following functional equation for the function $\varphi(u,v)$:

$$\varphi(\varphi(x,z), \varphi(y,z)) = \varphi(x,y). \tag{2}$$

Equation (2) is a natural consequence of the principle of phenomenological symmetry and, as it will be shown in what follows, this equation has a solution which is both physically and mathematically (geometrically) meaningful.

From the mathematical standpoint it seems to be of interest to compare all equations which can be obtained from (2) by permutations of variables in its left-hand side. Among these twelve equations the eight equations (including (2)) exist not reducible to each other:

$$\varphi(\varphi(y,z), \varphi(x,z)) = \varphi(x,y), \tag{3}$$
$$\varphi(\varphi(x,z), \varphi(z,y)) = \varphi(x,y), \tag{4}$$
$$\varphi(\varphi(z,y), \varphi(x,z)) = \varphi(x,y), \tag{5}$$
$$\varphi(\varphi(z,x), \varphi(y,z)) = \varphi(x,y), \tag{6}$$
$$\varphi(\varphi(y,z), \varphi(z,x)) = \varphi(x,y), \tag{7}$$
$$\varphi(\varphi(x,y), \varphi(z,z)) = \varphi(x,y), \tag{8}$$
$$\varphi(\varphi(y,x), \varphi(z,z)) = \varphi(x,y). \tag{9}$$