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INFINITESIMAL DEFORMATIONS OF SYMPLECTIC STRUCTURE WITH SINGULARITIES

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In the present paper we study infinitesimal deformations of a symplectic structure with singularities on a compact manifold. We apply our results to a symplectic structure with Martinet singularities [1], [2].

All the manifolds and mappings are assumed to be smooth. For a manifold M, we denote by $\Omega(M)$ the algebra of differential forms on M, and by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. For a vector field $V \in \mathfrak{X}(M)$, we denote by L_V the Lie derivative with respect to V.

1. Symplectic structure with singularities

Let ω_0 be a closed differential 2-form on $\mathbb{R}^{2n}$ such that $\Sigma_0 = \{x \in \mathbb{R}^{2n} \mid \det \omega_0(x) = 0\}$ is an embedded manifold, and Γ_0 be the pseudogroup consisting of local diffeomorphisms f of $\mathbb{R}^{2n}$ such that $f^*\omega_0 = \omega_0$.

A symplectic structure with singularities of type ω_0 on a $2n$-dimensional manifold M can be defined in two ways.

Definition 1. *A symplectic structure with singularities of type* ω_0 on a $2n$-dimensional manifold M is a Γ_0-structure on M, i.e., a maximal atlas whose transition functions lie in Γ_0.

Definition 2. A 2-form ω on a $2n$-dimensional smooth manifold M is called a *symplectic structure with singularities of type* ω_0 if for any point $p \in M$ there exist a neighborhood $U(p)$ and a diffeomorphism $\phi_p : U(p) \to \mathcal{O} \subset \mathbb{R}^{2n}$, where $\mathcal{O}$ is an open subset in $\mathbb{R}^{2n}$, such that $\omega|_{U(p)} = \phi_p^*(\omega_0)$.

These definitions are equivalent. Suppose that on M we have a Γ_0-atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$. Then $\phi_\alpha^*\omega_0|_{U_\alpha \cap U_\beta} = \phi_\beta^*\omega_0\big|_{U_\alpha \cap U_\beta}$. Therefore, the form $\omega \in \Omega^2(M)$ such that $\omega|_{U_\alpha} = \phi_\alpha^*\omega_0$ is correctly defined on M. It is evident that ω satisfies Definition 2. Now, suppose that a 2-form satisfying Definition 2 is given on M. Then the set of pairs $(U(p), \phi_p)$ is a Γ_0-atlas on M. It is clear that this correspondence between maximal Γ_0-atlases and 2-forms which satisfy Definition 2 is one-to-one.

The singular submanifold of a symplectic structure with singularities. Let a 2-form ω be a symplectic structure with singularities of type ω_0 on a $2n$-dimensional manifold M.

Lemma 1. *The set* $\Sigma = \{p \in M \mid \det \omega(p) = 0\}$ *is an embedded closed submanifold in* M, *and* $\dim \Sigma = \dim \Sigma_0$.

Proof. Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha : U_\alpha \to \mathcal{O}_\alpha)\}$ be the Γ_0-atlas corresponding to ω. It is clear that for any α, $U_\alpha \cap \Sigma = \phi_\alpha^{-1}(\mathcal{O}_\alpha \cap \Sigma_0)$. Since Σ_0 is a manifold embedded in $\mathbb{R}^{2n}$, $\Sigma_0 \cap \mathcal{O}_\alpha$ is a manifold embedded in $\mathcal{O}_\alpha$. And, since ϕ_α is a diffeomorphism, we have that $U_\alpha \cap \Sigma$ is a manifold embedded in U_α, hence Σ is a manifold embedded in M. □