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ASYMPTOTICS OF THE VOLUME FUNCTION OF SET OF LESSER VALUES OF A POLYNOMIAL IN $\mathbb{R}^n$

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Introduction

We investigate a function $V = V(\rho)$, which expresses the volume of the set

$$\Phi_\rho = \{x \in \mathbb{R}^n : f(x) < \rho\},$$

in which the values of a given polynomial f are less than ρ. The basic result states that for the elliptic polynomial f the function $V(\rho)$ can be expanded into the Laurent–Puiseux series converging for sufficiently large ρ, while the coefficient at the principal term of the series, which determines the asymptotics of the function of volume on infinity is expressed by an integral of a rational function over the space $\mathbb{R}^n$. It is shown that, in the general case (under absence of ellipticity), for calculation of the volume function, along with the integration procedure, one needs an additional procedure of passage to limit. We give examples for the calculation of the volume of the figures Φ_ρ.

1. General formulas

By general formulas we mean formulas for calculation of the volumes of figures of the form

$$\Phi_\rho = \{x \in \mathbb{R}^n : f(x) < \rho\}$$

in the space $\mathbb{R}^n$, where $f(x) = f(x_1, \dots, x_n)$ is a polynomial in n variables with real coefficients. We shall assume that the degree of f is even and $f(x) \geqslant 0$ in $\mathbb{R}^n$. In this case, for the volume of Φ_ρ we have

$$V(\Phi_\rho) = \int\limits_{f(x)<\rho} dx = \int\limits_{0<f(x)<\rho} dx_1 \wedge \dots \wedge dx_n. \tag{1}$$

By the Fubini theorem, integral (1) can be reduced to an iterated integral, in which we first integrate along level hypersurfaces $\{f(x) = t\}$, and then along the remaining variable t. To this end we start from an evident equality

$$\int\limits_{0<f(x)<\rho} dx_1 \wedge dx_2 \dots \wedge dx_n = \int\limits_{0<f(x)<\rho} \frac{dx_1 \wedge \dots \wedge dx_{n-1} \wedge f'_{x_n} dx_n}{f'_{x_n}}.$$

In addition, we note that

$$dx_1 \wedge \dots \wedge dx_{n-1} \wedge f'_{x_n} dx_n = dx_1 \wedge \dots \wedge dx_{n-1} \wedge df.$$