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BOUNDARY VALUES OF CAUCHY TYPE INTEGRAL ON A NONSMOOTH CURVE

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1. Introduction

Let Γ be a simple rectifiable curve on the complex plane. Then for any defined on Γ continuous function $f(t)$ the Cauchy type integral

$$C(\Gamma, f; z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)d\zeta}{\zeta - z} \tag{1}$$

exists and represents a function holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$. A traditional interest is attracted to the question of the existence and to the properties of the boundary values of this function, i. e., the limits $C^+(\Gamma, f; t) = \lim\limits_{z\to t} C(\Gamma, f; z)$ and $C^-(\Gamma, f; t) = \lim\limits_{z\to t} C(\Gamma, f; z)$, which arise within the approximation of z to a point $t \in \Gamma$ from the left and from the right, respectively. This interest is conditioned to a great extent by applications of integrals of Cauchy type to solving boundary value problems and singular integral equations (see [1], [2]).

As is well-known, integral (1) along a closed piecewise smooth curve Γ possesses continuous boundary values on Γ if its density $f(t)$ satisfies the Hölder condition

$$\sup\left\{\frac{|f(t') - f(t'')|}{|t' - t''|^\nu} : t', t'' \in \Gamma,\ t' \neq t''\right\} \equiv h_\nu(f, \Gamma) < \infty \tag{2}$$

with a certain indicator $\nu \in (0, 1]$. Thus result was known as far back as to Sokhotskiĭ, Harnack and Morera (see, e. g., [2], Chap. 1).

Later it was repeatedly refined and generalized.

In what follows we denote by $H_\nu(\Gamma)$ the Hölder space, i. e, a set of all functions given on Γ and satisfying condition (2).

One of important recent achievements in this domain is the theorem proved by Ye.M. Dyn'kin (see [3]) and also independently by T. Salimov (see [4]). This theorem supplies an estimate of the continuity module of integral (1) by a rectifiable (in general, nonsmooth) curve Γ via the continuity module of its density f and some other quantities which characterize the metric properties of Γ. The simplest consequences of this estimate are as follows:

(a) if $f \in H_\nu(\Gamma)$ for $\nu > 1/2$, then the boundary values $C^\pm(\Gamma, f; t)$ exist and are continuous without additional constraints upon the rectifiable curve Γ; this result along with the assertion about its unimprovability (see below) can be treated as a solution to the question on the possibility to transfer the Harnack–Morera–Sokhotskiĭ theorem to arbitrary nonsmooth rectifiable curves;

(b) if the curve Γ satisfies the condition $\theta_\Gamma(r) \asymp r$, where $\theta_\Gamma(r)$ is the maximal (with respect to $\zeta \in \Gamma$) summary length of arcs Γ lying in the disk $|z - \zeta| \leq r$, then the boundary values of integral

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