ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2012, Vol. 56, No. 6, pp. 52–56. © Allerton Press, Inc., 2012.
Original Russian Text © R.B. Salimov, 2012, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2012, No. 6, pp. 61–66.

The Behavior of a Singular Integral with the Hilbert Kernel Near a Point Where the Density of the Integral is Weakly Continuous

R. B. Salimov[1]

[1] *Kazan State University of Architecture and Engineering, ul. Zelyonaya 1, Kazan, 420043 Russia*

Received October 27, 2011

Abstract—We study the behavior of a singular integral with the Hilbert kernel near a point where the continuous density of the integral does not satisfy the Hölder condition and, as a result, the integral, possibly, diverges.

DOI: 10.3103/S1066369X12060072

Keywords and phrases: *singular integral, Hilbert kernel, weak continuity.*

We consider a singular integral (understood in the sense of the principal value) with the Hilbert kernel

$$I(\gamma_0) = \int_0^{2\pi} \varphi(\gamma) \cot \frac{\gamma - \gamma_0}{2} d\gamma, \tag{1}$$

where the density $\varphi(\gamma)$ is a continuous function defined on the segment $[0, 2\pi]$, $\gamma_0 \in [0, 2\pi]$, and $\varphi(0) = \varphi(2\pi)$. As is known ([1], pp. 18, 46), this integral exists at a point γ_0, if the function $\varphi(\gamma)$ satisfies at this point the Hölder condition (condition H). If the density $\varphi(\gamma)$ satisfies condition H in a certain part of the segment $[0, 2\pi]$, then by virtue of the Plemelj–Privalov theorem (ibid., pp. 58, 61) the integral $I(\gamma_0)$ satisfies condition H in any segment located inside this part. The behavior of integral (1) in the case, when the density $\varphi(\gamma)$ has an integrable singularity, is described by N. I. Muskhelishvili (ibid., pp. 95, 160; see also the paper [2]). The connection between the modulus of continuity of the singular integral and its density is considered in the paper [3].

If the function $\varphi(\gamma)$ is continuous at some point $\gamma = c$ of the interval $(0, 2\pi)$ but does not satisfy condition H, then integral (1) can diverge at this point. Therefore, in this case one has to study the behavior of integral (1) for $\gamma_0 \to c$.

We perform this study under the following restrictions. Let c^- and c^+ be given values sufficiently close to the point c, and $0 < c^- < c < c^+ < 2\pi$. We assume that the function $\varphi(\gamma)$ has a continuous derivative $\varphi'(\gamma)$ everywhere in the segment $[0, 2\pi]$ excluding the point c, $\varphi'(\gamma) \to \infty$ for $\gamma \to c$, and, in addition, the derivative $\varphi'(\gamma)$ is monotonic and does not vanish in each of intervals (c^-, c) and (c, c^+).

We assume also that the continuous function $\varphi(\gamma)$ is weakly continuous at the point c in the following sense: For an arbitrarily small positive constant α the ratio

$$|\varphi(\gamma) - \varphi(c)|/|\gamma - c|^\alpha, \quad 0 \le \gamma \le 2\pi, \quad \gamma \ne c,$$

is unbounded in a neighborhood of the point c. The weak continuity of the density $\varphi(\gamma)$ is also representable in the form

$$\int_0^\pi \frac{\omega(\delta, \varphi)}{\delta} d\delta = \infty$$

proposed in [3], where $\omega(\delta, \varphi)$ is the modulus of continuity of the density $\varphi(t)$ with the step δ. Under this condition integral (1) can be unbounded.

By integrating by parts (cf. [4]) we present integral (1) in the form

$$I(\gamma_0) = \int_0^{2\pi} \varphi'(\gamma) \Big(- \ln \sin^2 \frac{\gamma - \gamma_0}{2} \Big) d\gamma, \quad \gamma_0 \ne c. \tag{2}$$