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On an Extreme Point Conjecture for Concave Functions

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Abstract—Let CO(A), $A \in (1;2]$, denote the family of concave univalent functions in the unit disk $\mathbb{D}$ with opening angle at infinity bounded by πA. We prove a weak form of a conjecture on the extreme points of clco CO(A) from the paper in Indian J. Math. **50**, 339–349 (2008).

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1. INTRODUCTION AND THE MAIN RESULT

Let $\mathcal{H}$ denote the class of functions holomorphic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the topology of uniform convergence on compact subsets of $\mathbb{D}$. It is known that $\mathcal{H}$ is a complete, metrizable, locally convex linear topological space. The systematic applications of linear methods for the study of extreme points of subclasses of $\mathcal{H}$ is rather more recent. Let $\mathcal{F} \subset \mathcal{H}$. A function f is an extreme point of $\mathcal{F}$ if $f \in \mathcal{F}$ and f is not a proper convex combination of two distinct functions in $\mathcal{F}$. We use co $\mathcal{F}$ to denote the convex hull of $\mathcal{F}$ and clco $\mathcal{F}$ to denote the closed convex hull of $\mathcal{F}$. Two important issues concerning a given compact family $\mathcal{F}$ are to determine clco $\mathcal{F}$ and identify geometric and analytic properties of functions in the set of extreme points of $\mathcal{F}$.

Let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ with the standard normalization $f(0) = f'(0) - 1 = 0$, and $\mathcal{S}$ be the class of functions in $\mathcal{A}$ that are univalent (schlicht) in $\mathbb{D}$. In this paper, we are mainly concerned with extreme points of the closed convex hulls of the families CO(A), $A \in (1, 2]$, of concave schlicht functions with opening angle at infinity is less than or equal to πA. These families have been introduced in [1] (cf. [2] and [3]). A function $f \in \mathcal{A}$ is said to belong to CO(A), $A \in (1, 2]$, if it satisfies the following conditions:

(i) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex, and satisfies the additional normalization condition: $f(1) = \infty$;

(ii) the opening angle of $f(\mathbb{D})$ at the point at infinity is less than or equal to πA, $A \in (1, 2]$. This means that the smallest wedge shaped region containing $f(\mathbb{D})$ has an opening angle less than or equal to πA.

In [1], it has been proved that f is a member of the family CO(A) if and only if there exists a holomorphic function $\varphi : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\frac{f''(z)}{f'(z)} = \frac{A+1}{1-z} + \frac{(A-1)\varphi(z)}{1+z\varphi(z)}, \quad z \in \mathbb{D}. \tag{1}$$

This characterization was proved in [4] in the case $A = 2$ and in [1] in the case $A \le 2$.

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