Jordan groups

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Definition (P., 2010)

A group G is called *Jordan* if all its finite subgroups are "almost Abelian" in the following sense: there exists a finite list of finite groups such that every finite subgroup of G is an extension of an Abelian group by a group taken from this list.

Reformulation:

Given a group G, let

$$J(G) := \sup_F \min_A [F : A],$$

where F runs over all finite subgroups of G and A runs over all normal Abelian subgroups of F. If $J(G) < \infty$, then G is called a *Jordan group*, and $J(G)$ is called the *Jordan constant of G*.

Remark

In this definition, the assumption of normality *can be dropped*: let

$$j(G) := \sup_F \min_A [F : A],$$

where F runs over all finite subgroups of G and A runs over all (not necessarily normal!) Abelian subgroups of F.

Proposition

G is Jordan if and only if $j(G) < \infty$.

If G is Jordan, $j(G)$ is called the *weak Jordan constant* of G; in this case,

$$j(G) \leqslant J(G) \leqslant j(G)^2.$$

Two main sources of the above definition:

(1) *First source: Jordan's theorem on linear groups*

Theorem (C. Jordan, 1878)

For every field k of characteristic 0 and every positive integer n, the group $\mathrm{GL}_n(k)$ is Jordan.

Corollary (on linear groups)

For every field k of characteristic 0 and every positive integer n, any subgroup of $\mathrm{GL}_n(k)$ is Jordan.

This and Chevalley's theorem of 1958 yield:

Corollary (on affine algebraic groups)

Every affine algebraic group over a field of characteristic 0 is Jordan.

Computing $J(n) := J(\mathrm{GL}_n(\mathbb{C}))$:

Upper bounds:

Theorem (I. Schur, 1911)

$$J(n) \leqslant (\sqrt{8n}+1)^{2n^2} - (\sqrt{8n}-1)^{2n^2}.$$

Theorem (H. Blichfeldt, 1917)

$$J(n) < n!\, 6^{(n-1)(\pi(n+1)+1)},$$

where $\pi(s)$:= *number of primes not exceeding* s.

Lower bound:

$\mathrm{GL}_n(\mathbb{C}) > \mathrm{Sym}_{n+1}$, and the only proper normal subgroup of Sym_{n+1} for $n \geqslant 2, n \neq 3$ is Alt_{n+1}. Hence

$$(n+1)! \leqslant J(n) \quad \text{for every } n \geqslant 4.$$

Exact value:

Theorem (M. J. Collins, 2007)

$$J(n) = \begin{cases} (n+1)! \;\; \textit{if } n \geqslant 71 \textit{ and } n = 63, 65, 67, 69, \\ 69^r r! \;\; \textit{if } n = 2r \textit{ or } 2r+1 \textit{ and either } 20 \leqslant n \leqslant 62 \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{or } n = 64, 66, 68, 70, \\ 60, 360, 25920, 25920, 6531840 \;\; \textit{for } n = 2, 3, 4, 5, 6. \end{cases}$$

The value of $J(n)$ for $7 \leqslant n \leqslant 19$ see in M. J. Collins, On Jordan's theorem for complex linear groups, J. Group Theory **10** *(2007), 411–423.*

(2) *Second source: Serre's theorem on the Cremona group of rank 2*

Definition

Let $k(x_1, \ldots, x_n)$ be the field of rational functions in variables $x_1, \ldots, x_n$ with the coefficients in a field k. The group $\mathrm{Cr}_n(k) := \mathrm{Aut}_k k(x_1, \ldots, x_n)$ is called the *Cremona group* over k of rank n.

Geometric meaning:

$\mathrm{Cr}_n(k)$ is the group of birational k-selfmaps of any n-dimensional algebraic variety X rational over k (e.g., affine space $\mathbb{A}^n$ or projective spave $\mathbb{P}^n$).

$\underline{n = 1}$:

$\mathrm{Cr}_1(k)$ consists of all fraction-linear transformations

$$x_1 \mapsto \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \alpha, \beta, \gamma, \delta \in k, \quad \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0.$$

Hence $\mathrm{Cr}_1(k) = \mathrm{PGL}_2(k)$ is Jordan.

$\underline{n \geqslant 2}$:

Theorem (D. Cerveau, J. Deserti, 2009)

For $n \geqslant 2$ and $\operatorname{char} k = 0$, the group $\mathrm{Cr}_n(k)$ is nonlinear.

Convention:

For simplicity,

everywhere below k is an algebraically closed field of characteristic 0,

and $\mathrm{Cr}_n(k)$ is denoted by Cr_n.

Classification of finite subgroups in Cr_n up to conjugacy:

$n = 1$: Classically known (F. Klein, 1884). The complete list consists of a single conjugacy class for each of the following groups: cyclic groups of orders $1, 2, 3, \ldots$, dihedral groups of orders $4, 6, 8, \ldots$, rotation groups of tetrahedron (Alt_4), octahedron (Sym_4), and icosahedron (Alt_5).

$n = 2$: Obtained in 2009 by I. Dolgachev and V. Iskovskikh. The answer is given by the complicated lists. There are infinitely many conjugacy classes and even parameter-depending families of such classes (for cyclic groups).

$n = 3$: Classification is hardly possible.

This leads to a change of the viewpoint: the problem of classifying conjugacy classes is replaced by that of finding general qualitative properties of all finite groups in the considered families. The property of being Jordan is the example of such a property.

The first result shaping this viewpoint was obtained by J.-P. Serre who proved the following

Theorem (J.-P. Serre, 2008)

The Cremona group Cr_2 *is Jordan.*

Clearly, $J(\mathrm{Cr}_1) = |\mathrm{Alt}_5| = 60$ and $j(\mathrm{Cr}_1) = 12$.

At the moment (June 2019), $j(\mathrm{Cr}_2)$ and $j(\mathrm{Cr}_3)$ are known as well:

Theorem (Y. Prokhorov, C. Shramov, 2017)

$j(\mathrm{Cr}_2) = 288$;
$j(\mathrm{Cr}_3) = 10368$.

Other Jordan groups:

Definition (P., 2010)

Given a group H, put

$$b_H := \sup_F |F|,$$

where F runs over all finite subgroups of H. If $b_H \neq \infty$, then H is called a *bounded group*.

Theorem (P., 2018)

Let G be a finite-dimensional real Lie group such that the component group G/G^0 is bounded. Then G is Jordan and

$$J(G) \leqslant b_{G/G^0} J(\dim G(2^{\dim G} + 10))^{b_{G/G^0}}.$$

Corollary (on (not necessarily affine) algebraic groups)

Every (not necessarily affine) algebraic group G is Jordan and

$$J(G) \leqslant [G : G^0] J(\dim G(2^{2\dim G+1} + 20))^{[G:G^0]}.$$

Corollary

For every finite-dimensional real Lie group G whose component group G/G^0 is bounded, the set of isomorphism classes of all finite simple groups embeddable in G is finite.

Jordan groups: research programs (P., 2010)

Problem (A)

Describe algebraic varieties X, for which the group $\mathrm{Aut}(X)$ is Jordan.

Problem (B)

Describe algebraic varieties X, for which the group $\mathrm{Bir}(X)$ is Jordan.

Reformulations:

Problem B is equivalent to

Problem (B')

Describe finitely generated field extensions K/k such the group $\mathrm{Aut}_k(K)$ is Jordan.

For affine X, Problem A is equivalent to

Problem (A')

Describe (commutative) finitely generated k-algebras A without nilpotent elements such the group $\mathrm{Aut}_k(A)$ is Jordan.

Results on Problem (B)

(1) *Small dimensions*:
In dimensions 1, 2, 3 the situation is now completely clear:

Theorem (Yu. Zarhin, 2010)

Let X be a product of an elliptic curve and a projective line. Then the group $\mathrm{Bir}(X)$ is non-Jordan.

Reformulation:

Let L/k be a field extension of transcendence degree 1 and of genus 1, and let K be a field of rational functions in one variable with the coefficients in L. Then the group $\mathrm{Aut}_k(K)$ is non-Jordan.

It turns out that in dimension < 3 it is the only type of exceptions:

Theorem (P., 2010)

For any irreducible algebraic variety X of dimension < 3, the group $\mathrm{Bir}(X)$ is non-Jordan if and only if X is birationally isomorphic to the product of an elliptic curve and a projective line.

Reformulation:

For every finitely generated field extension K/k of transcendence degree < 3, the group $\mathrm{Aut}_k(K)$ is non-Jordan if and only if K is a field of rational functions in one variable with the coefficients in a field extension of k of transcendence degree 1 and genus 1.

The following provides complete answer in dimension 3:

Theorem (Yu. Prokhorov, C. Shramov, 2017)

For any irreducible 3*-dimensional algebraic variety* X*, the group Bir(X) is non-Jordan if and only if* X *is birationally isomorphic either to* $E \times \mathbb{P}^2$*, where* E *is an elliptic curve, or to* $S \times \mathbb{P}^1$*, where* S *is one of the following:*

- *an Abelian surface;*
- *a bielliptic surface;*
- *a surface of Kodaira dimension* 1 *such that the Jacobian fibration of the pluricanonical fibration* $S \to B$ *is locally trivial in Zariski topology.*

(2) *Cremona groups of arbitrary rank*:

In 2018 the result of J.-P. Serre was extended to the Cremona groups of arbitrary rank:

Theorem (Yu. Prokhorov and C. Shramov, 2016; C. Birkar, 2018)

The Cremona group Cr_n *is Jordan for every* n*.*

Remark

Yu. Prokhorov and C. Shramov gave a conditional proof modulo the so-called BAB-conjecture. The latter was then proved by C. Birkar, thereby completing the proof.

Application to Serre's question:

Question (J.-P. Serre, 2009)

Does there exist a finite group which is not embeddable in Cr_3*?*

The answer is a special case of the following:

Theorem (P., 2018)

There exists an integer $b_{n,k}$, depending on n and k, such that every product of groups $G_1 \times \cdots \times G_s$, each of which contains a finite non-Abelian subgroup, is nonembeddable in the Cremona group Cr_n if $s > b_{n,k}$.

Corollary

For every prime integer p, there exists a non-Abelian finite p-group nonembeddable in Cr_n.

Results on Problem (A)

At the moment (June 2019), the answer to the following question is *unknown*:

Question

Are there algebraic varieties X such that the group $\mathrm{Aut}(X)$ is non-Jordan?

For *affine* X, it admits the following

Reformulation:

Are there finitely generated (commutative) k-algebras A without nilpotent elements such the group $\mathrm{Aut}_k(A)$ is non-Jordan?

However, for many types of X, it is now proved that $\mathrm{Aut}(X)$ is Jordan:

(1) Small dimensions:

Theorem (T. Bandman and Yu. Zarhin, 2014)

For every algebraic variety X of dimension < 3, the group $\mathrm{Aut}(X)$ is Jordan.

Remark

In fact, the key part is a proof that if X is birationally isomorphic to a product of an elliptic curve and a projective line, then the group $\mathrm{Aut}(X)$ is Jordan. The rest is the corollary of the above theorem that for all algebraic surfaces Y, which are not birationally isomorphic to such a product, the group $\mathrm{Bir}(Y)$ is Jordan.

(2) Nonsmooth algebraic varieties:

Definition

A point x of an algebraic variety X is called a *vertex*, if for every point $y \in X$,

$$\dim(\text{tangent space of } X \text{ at } x) \geqslant \dim(\text{tangent space of } X \text{ at } y)$$

Example

If X is a closed cone in $\mathbb{A}^n$ such that for every point $x \in X$, $x \neq 0$, the line passing through x and 0 lies in X, then 0 is a vertex of X.

Theorem (P., 2011)

If X has only finitely many vertices, then the group $\mathrm{Aut}(X)$ is Jordan.

Corollary

The automorphism group of every nonsmooth algebraic variety with only finitely many singular points is Jordan.

(3) Projective algebraic varieties

Theorem (S. Meng and D.-Q. Zhang, 2015)

Let X be a projective algebraic variety. Then the group $\mathrm{Aut}(X)$ is Jordan.

(3) Affine algebraic varieties

Definition (P., 2010)

An algebraic variety X is called *toral* if it is isomorphic to a closed subvariety of $\underbrace{k^\times \times \cdots \times k^\times}_{n}$ for some n (depending on X).

Reformulation:

Toral varieties are precisely affine varieties whose coordinate algebras are generated by invertible elements.

Theorem (P. 2010)

Let X be a toral algebraic variety. Then the group $\mathrm{Aut}(X)$ is Jordan.

<u>Reformulation</u>:

For every (commatative) k-algebra A without nilpotent elements generated by finitely many invertible elements, the group $\mathrm{Aut}_k(A)$ is Jordan.

Other settings: Differentiable manifold

Problem

Describe differentiable manifolds M such that the group $\mathrm{Diff}(M)$ of all self-diffeomorphisms $M \to M$ is Jordan.

(1) Noncompact manifolds:

Theorem (P., 2014)

For every integer $n \geqslant 4$, there exists a simply connected noncompact smooth oriented n-dimensional topological manifold M_n such that the group $\mathrm{Diff}(M_n)$ contains an isomorphic copy of every finitely presented group, and this copy is a discrete transformation group of M_n acting freely.

Corollary (P., 2014)

For every integer $n \geqslant 4$, there exists a simply connected noncompact smooth oriented n-dimensional topological manifold X_n such that the group $\mathrm{Diff}(X_n)$ contains a freely acting on X_n isomorphic copy of every finite group.

Remark

Explicit construction: X_n is the universal cover of the connected sum of 4 copies of $S^1 \times S^3$, to which 14 "handles" $D^2 \times S^2$ are successively glued.

Corollary (P., 2014)

For every integer $n \geqslant 4$, there exists a simply connected noncompact smooth oriented n-dimensional topological manifold X_n such that the group $\mathrm{Diff}(X_n)$ is non-Jordan.

On the other hand, there are many important classes of noncompact differentiable manifolds M, for which the group $\mathrm{Diff}(M)$ is Jordan. In particular,

Theorem (I. Mundet i Riera, 2014)

The group $\mathrm{Diff}(\mathbb{R}^n)$ is Jordan for every n.

Corollary

The group $\mathrm{Aut}(\mathbb{A}^n)$ *is Jordan for every* n.

Reformulation:

Corollary

For any polynomial ring $P_n = k[x_1, \ldots, x_n]$, *the group* $\mathrm{Aut}_k(P_n)$ *is Jordan.*

(2) Compact manifolds:

Conjecture (É. Ghys, 1997)

For every connected compact smooth manifold M, the group $\mathrm{Diff}(M)$ is Jordan.

A counterexample was constructed in

Theorem (B. Csikós, L. Pyber, E. Szabó, 2014)

The group $\mathrm{Diff}(S^2 \times S^1 \times S^1)$ *is non-Jordan.*

Remark

The proof is based on Zarhin's idea, which was used for proving that $\mathrm{Bir}(C \times \mathbb{P}^1)$ is non-Jordan.

Other settings: Complex spaces

Problem (CA)

Describe complex manifolds (more generally, complex spaces) M such that the group $\mathrm{Aut}(M)$ of all biholomorphic self-maps $M \to M$ is Jordan.

Problem (CB)

Describe complex manifolds M such that the group $\mathrm{Bim}(M)$ of all bimeromorphic self-maps $M \to M$ is Jordan.

(1) *Small dimensions*:

Theorem (Yu. Prokhorov, C. Shramov, 2019)

Let X be a connected compact manifolds of dimension < 3. Then the group $\mathrm{Aut}(X)$ if Jordan.

Theorem (Yu. Prokhorov, C. Shramov, 2019)

Let X be a connected compact complex surface. Then the group $\mathrm{Bim}(X)$ if Jordan if and only if X is not bimeromorphically isomorphic to the product of an elliptic curve and a projective line.

(2) *The groups* $\mathrm{Aut}(M)^0$:

Theorem (P., 2018)

For every compact complex space M, the identity component $\mathrm{Aut}(M)^0$ of the group $\mathrm{Aut}(M)$ is Jordan.

Theorem (P., 2018)

For every compact complex manifold M of complex dimension n, the following inequality holds

$$J(\mathrm{Aut}(M)^0) \leqslant J((2n^2+n)(2^{2n^2+n}+10)).$$

In the above theorems, the assumption of compactness cannot be dropped:

Theorem (P., 2018)

There is a 3-dimensional simply connected noncompact complex manifold C such that

(i) *the group* $\mathrm{Aut}(C)$ *contains an isomorphic copy of every finitely presentable* (*in particular, every finite*) *group;*

(ii) *every such copy is a discrete transformation group of C acting freely.*

Corollary

The group $\mathrm{Aut}(C)$ *is non-Jordan.*

However, for noncompact complex manifolds M of several important types, the groups $\mathrm{Aut}(M)^0$ (or even $\mathrm{Aut}(M)$) are Jordan:

Theorem (P., 2018)

For every connected complex manifold H hyperbolic in the sense of Kobayashi and of complex dimension n, the group $\mathrm{Aut}(H)^0$ is Jordan and

$$J(\mathrm{Aut}(M)^0) \leqslant J((2n+n^2)(2^{2n+n^2}+10)).$$

Theorem (P., 2018)

For every strongly pseudoconvex bounded domain M with smooth boundary in $\mathbb{C}^n$, the group $\mathrm{Aut}(M)$ of all biholomorphic transformations of M is Jordan.

Corollary

For every strongly pseudoconvex bounded domain M with smooth boundary in $\mathbb{C}^n$, the set of isomorphism classes of all finite simple groups of biholomorphic transformations of M is finite.

Other settings: Riemannian manifolds

Theorem (P., 2018)

For every connected n-dimensional Riemannian manifold M, the identity component $\mathrm{Iso}(M)^0$ of the isometry group $\mathrm{Iso}(M)$ of M is Jordan and

$$J(\mathrm{Iso}(M)^0) \leqslant J((n^2+n)(2^{(n^2+n-2)/2}+5)).$$

If M is compact, then the group $\mathrm{Iso}(M)$ is Jordan.

Remark

In the last theorem, the group $\mathrm{Iso}(M)^0$ cannot be replaced by $\mathrm{Iso}(M)$: there are noncompact 2-dimensional Riemannian manifolds M such that the group $\mathrm{Iso}(M)$ is non-Jordan.