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Reduction of Solution Distance in Nonconvex Smooth Extremal Problems

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1. Consider the following nonlinear optimization problem:

$$\min_{x\in Q} \varphi(x), \tag{1}$$

where $\varphi : H \to \mathbb{R}$ is a twice continuously differentiable functional, Q is a convex closed set of a real Hilbert space H. Assume that the second derivative φ'' satisfies the Lipschitz condition

$$\|\varphi''(x) - \varphi''(y)\| \le L\|x - y\| \;\; \forall x, y \in H. \tag{2}$$

Hereinafter the symbol $\|\cdot\|$ is used for the uniform notation of norms of various spaces. The functional φ is not necessarily convex. If a point $x^* \in Q$ provides a local minimum in problem (1), then the following inequality is true:

$$(\varphi'(x^*), x^* - z) \le 0 \;\; \forall z \in Q. \tag{3}$$

We seek for an approximation of such a point. In particular, we seek for an approximation of an element $x^* \in Q$ which realizes a local or global minimum in (1). In what follows, we assume that the desired point exists.

Several problems can be reduced to the form (1); for example, problems, implying the search of solutions and quasisolutions of operator equations $F(x) = 0\,(x \in H)$ with a smooth nonlinear operator $F : H \to Y$, acting into a Hilbert space Y. To this end, one usually puts

$$\varphi(x) = \frac{1}{2}\|F(x)\|^2, \quad x \in H. \tag{4}$$

In applications to inverse problems of mathematical physics, the case, when the operator F is irregular, is rather typical. In this case linear operators $F'(x)$, $F'^*(x)F'(x)$ are not continuously invertible at points x from a neighborhood of x^*, and the corresponding functional (4), as a rule, is nonconvex. Many nonconvex problems (1) with functionals of the general form occur in the optimal control theory (e.g., [1], P. 283).

If the functional φ is nonconvex, then the problem on the numeric approximation of points x^*, which either provide a local or global minimum in (1), or satisfy the necessary condition (3), becomes rather laborious. This laboriousness is connected with the fact that the objective functional can be multiextremal. Moreover, in a nonconvex case one cannot guarantee the reduction of the distance to the desired solution x^* even if the functional monotonically decreases along the constructed iterative sequence. It is well-known that in an infinite-dimensional space H the sequence which minimizes a strictly convex functional can tend in the norm to infinity. Constructing iterative solution methods for irregular operator equations, one elaborated two essentially different approaches. Theoretically they allow one to overcome the mentioned difficulties. According to one approach (e. g., [2], pp. 278–279; [3], pp. 83–87), it is required that near the solution the operator F satisfies additional structural conditions of a generalized regularity. Note that in most applied inverse problems connected with irregular operator equations, the constructive validation of these conditions represents essential technical difficulties.

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