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ε-KERNEL OF BOUNDED SET IN A SPECIAL METRIC SPACE

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In this article we prove that the mapping which sends each nonempty bounded set in a special metric space to the ε-kernel of this set, is uniformly continuous with respect to the Hausdorff metric. This property generalizes the property of strong stability of Chebyshev center of bounded set in a uniformly convex Banach space (see [1]).

1. Preliminary. Definitions and Theorems

Let us recall definitions and notions for a metric space (X, ρ). The Chebyshev radius of a bounded set $M \subset X$ is the number (see [2])

$$R(M) = \inf_{y\in X} \sup_{x\in M} \rho(x, y).$$

A point $z \in X$ such that

$$\sup_{x\in M} \rho(x, z) = \inf_{y\in X} \sup_{x\in M} \rho(x, y)$$

is called the Chebyshev center of M (ibid.). The Hausdorff metric on the set $B[X]$ of all nonempty bounded closed sets in the space X is the metric (see [3], p. 223)

$$\delta(M, N) = \max\{\sup_{x\in M} \rho(x, N);\ \sup_{y\in N} \rho(y, M)\}$$

for $M, N \in B[X]$. Evidently, on the space $B(X)$ of all nonempty bounded sets in X, this expression defines the Hausdorff pseudometric.

Let $\varepsilon > 0$, $M \in B(X)$, $B[x, r]$ ($B(x, r)$) be a closed (open) ball of radius $r > 0$ with the center $x \in X$, and $B(M, r) = \bigcup\{B(x, r) : x \in M\}$. We set

$K_\varepsilon(M) = \{y \in X : \sup_{x\in M} \rho(x, y) \le R(M) + \varepsilon\}$,

$\varepsilon(M) = \bigcap\{B[x, R(M) + \varepsilon] : M \subset B[x, R(M) + \varepsilon]\}$,

$k_\varepsilon(M) = \bigcap\{B[x, r(x, M) + \varepsilon] : x \in \varepsilon(M)\}$, where $r(x, M) = (\rho(x, M) + \sup_{y\in M} \rho(x, y))/2$.

Definition. The ε-kernel of a set M is the set $K_\varepsilon(M)$, where $M \in B(X)$ and $0 < \varepsilon < R(M)$.

Let us list elementary properties of the sets above, which can be immediately obtained from the definitions.

1. For each $M \in B(X)$, $K_\varepsilon(M)$, $\varepsilon(M)$ lie in $B[X]$, and $\varepsilon(M) = \bigcap\{B[x, R(M)+\varepsilon] : x \in K_\varepsilon(M)\}$, $K_\varepsilon(M) = \bigcup\{y \in X : M \subset B[y, R(M)+\varepsilon]\} = \bigcup\{y \in X : \varepsilon(M) \subset B[y, R(M)+\varepsilon]\} = \bigcap\{B[x, R(M)+\varepsilon] : x \in \varepsilon(M)\}$.
2. If $\alpha \le \varepsilon$, then $K_\alpha(M) \subset K_\varepsilon(M)$.
3. If $M \in B(X)$ has a Chebyshev center, then $\bigcap\{K_\varepsilon(M) : \varepsilon > 0\}$ is the set of all Chebyshev centers of M.

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