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THE TANGENT SPACE IN THE SENSE OF BUSEMANN

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We study metric properties of the tangent space for a metric space more general than a differentiable G-space of Busemann. It is established that the metric on the tangent space at an arbitrary point of a space of nonpositive curvature in the sense of Busemann (a differentiable metric space in the sense of Busemann) is intrinsic. It is proved that the tangent space at an arbitrary point of a locally complete differentiable metric space in the sense of Busemann is complete, and that the tangent space at an arbitrary point of a locally compact space of nonpositive curvature in the sense of Busemann is finitely compact geodesic space.

1. Definitions and theorems

Consider a metric space (X, ρ) with distinguished family of segments S (a segment $[x, y]$ with ends $x, y \in X$ is a continuous curve connecting x and y whose length is equal to the distance $\rho(x, y)$ between x and y ([1], p. 42)) which satisfies the following properties.

A. Each subsegment of an arbitrary segment from S belongs to S.

B. For each $p \in X$, there exists a positive real number $r(p)$ such that, for any x and y from the open ball $B(p, r(p))$, there exists a unique segment $[x, y] \in S$.

In what follows we will use the following notation and definitions. $\mathbb{R}_+$ is the set of nonnegative real numbers; $xy = \rho(x, y)$. $B[x, r]$ ($B(x, r)$, $S(x, r)$) is a closed ball (open ball, sphere) of radius $r > 0$ centered at $x \in X$. For $\lambda \in [0, 1]$ and $x, y \in X$, $\omega_\lambda[x, y]$ is the point of the segment $[x, y] \in S$ such that $x\omega_\lambda[x, y] = \lambda xy$.

A space (X, ρ) satisfying conditions A and B will be called differentiable at $p \in X$ in the sense of Busemann if, for each $\varepsilon > 0$, there exists $\delta \in (0, r(p))$ such that $|\omega_\lambda[p, x]\omega_\lambda[p, y] - \lambda xy| \le \varepsilon\lambda xy$ for any $\lambda \in [0, 1]$, $x, y \in B(p, \delta)$ ([1], p. 293; [2], p. 21).

A space (X, ρ) satisfying conditions A and B will be called a space of nonnegative curvature in the sense of Busemann if $2\omega_{1/2}[z, x]\omega_{1/2}[z, y] \le xy$ for any $x, y, z \in B(p, r(p))$ ([1], p. 304; [3], p. 63).

A metric space X is called a space with intrinsic metric if, for any $x, y \in X$, $\varepsilon > 0$, there exists a finite sequence of points $z_0 = x, z_1, \cdots, z_k = y$ such that $z_i z_{i+1} < \varepsilon$ $(0 \le i \le k-1)$ and $z_0 z_1 + \cdots + z_{k-1} z_k < xy + \varepsilon$ [4].

A space (X, ρ) is called geodesic if any two its points can be connected by a segment [5] (in the first version of this definition, X was assumed to be a complete metric space [6]).

A space (X, ρ) is called a finitely compact metric space if each its bounded closed subset is compact ([1], p. 18).

A mapping f from a metric space (X, ρ) to a metric space (Y, d) is called a bi-Lipschitz mapping if there exists a constant $\lambda \ge 1$ such that $\rho(x, y)/\lambda \le d(f(x), f(y)) \le \lambda\rho(x, y)$ for any $x, y \in X$. Metric spaces X and Y are said to be bi-Lipschitz equivalent if there exists a surjective bi-Lipschitz mapping $f : X \to Y$ ([7], p. 269).

Below we formulate the results obtained in this paper.