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SOME SOLUTIONS OF THE PROBLEM ON DECAY OF GAP IN NON-NEWTON LIQUID'S DYNAMICS

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1. The dynamics of film flows of a nonlinearly viscous liquid with an exponential rheological law in the one-dimensional approximation is described by the equation (see [1]–[3])

$$\frac{\partial l}{\partial t} = \frac{\partial q}{\partial x}, \quad t > 0, \quad -\infty < x < \infty, \tag{1.1}$$

where l is a dimensionless thickness of the film, q is a dimensionless flow of liquid

$$q = \operatorname{sgn} \frac{\partial l}{\partial x} \left(\frac{l^{2+n}}{n+2} \left| \frac{\partial l}{\partial x} \right|^n \right), \quad n > 1, \tag{1.2}$$

t is dimensionless time, and the constant n is defined by the rheological law.

By the problem on decay of gap for equation (1.1) we understand the requirement to find its nonincreasing nonnegative solution $l(t,x)$ continuous on the whole x axis along with the flow q, which satisfies the initial condition

$$l(0,x) = l_0(x); \quad l_0(x) = 1 \ \ (x \le 0), \quad l_0(x) = 0 \ \ (x > 0). \tag{1.3}$$

The importance of the problem on decay of gap in the construction of a solution to the Cauchy problem with discontinuous initial data is well-known (see [4]). In Item 2 we construct the unique automodelling solution of problem (1.1)–(1.3), which is the basic result of this article. In Item 3, using the results obtained in Item 2, we consider automodelling discontinuous solutions of equation (1.1); to this end we slightly modify condition (1.3). In what follows we use the notation

$$\alpha = \frac{1}{n+1}, \quad \beta = \frac{1}{n+2}, \quad \gamma = \frac{n+1}{2n+1}, \quad \delta = \frac{1}{2n+1}, \quad \lambda = \frac{1}{n-1}, \quad \mu = \frac{1}{n-2}.$$

2. In this Item for the problem on decay of gap for equation (1.1) we construct a nonincreasing nonnegative and continuous along with the flow $q = -\beta l^{n+2}(-l_x)^n$ solution of the equation

$$l_t = -(\beta l^{n+2}(-l_x)^n)_x, \tag{2.1}$$

which satisfies the initial condition (1.3). Equation (2.1) together with conditions (1.3) admits a group of expansions with the operator

$$X = t\partial_t + \alpha x \partial_x. \tag{2.2}$$

A solution of the problem on decay of gap for equation (2.1) will be constructed in the form $l(t,x) = V(\xi)$, where $\xi = xt^{-\alpha}$ is the invariant of the group (2.2). From equation (2.1) and initial condition (1.3) we find for a nonnegative decreasing function V the nonlinear equation

$$\alpha \xi V_\xi = (\beta V^{n+2}(-V_\xi)^n)_\xi \tag{2.3}$$

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and the boundary conditions

$$V(\infty) = 0, \tag{2.4}$$

$$V(-\infty) = 1. \tag{2.5}$$

The continuity of the flow implies the continuity of the function $q_1 = \beta V^{n+2}(-V_\xi)^n$ for $|\xi| < \infty$. We consider equation (2.3) first in the halfplane $\xi > 0$. To decrease the order of equation (2.3) we set

$$V = \xi^\gamma \varphi(\xi), \quad \psi = \xi \varphi'_\xi. \tag{2.6}$$

We will have

$$V'_\xi = \xi^{-n\delta} W(\varphi, \psi), \tag{2.7}$$

$W = \psi + \gamma\varphi$. After substitution of relations (2.6), (2.7) into equation (2.3) we find that the solutions of equation (2.3), which differ from the solutions $V = \text{const}$, can be obtained with the help of (2.6) from the first order equation

$$\frac{d\psi}{d\varphi} = -\frac{P_1(\varphi, \psi)}{Q_1(\varphi, \psi)}, \tag{2.8}$$

where

$$P_1 = -\frac{\alpha}{\beta} + \varphi^{n+1}(-W)^{n-2}[\gamma n \varphi \psi + \delta(3n+2)\varphi W + \tfrac{1}{\beta}\psi W], \quad Q_1 = n\varphi^{n+2}\psi(-W)^{n-2}.$$

Every solution $\psi = \psi(\varphi)$ of equation (2.8) generates a one-parametric family of solutions of the equation with separable variables $\psi = \xi\varphi'_\xi$, and afterwards by means of the first of relations (2.6) also a one-parametric family of solutions of equation (2.3).

First of all we construct in the halfplane $\xi > 0$ a nonincreasing solution V_1 of equation (2.3), which satisfies the boundary condition (2.4) and the condition of continuity of the flow q. By virtue of (2.7) and the condition that solution V_1 does not increase, equation (2.8) must be considered in the sector I, which is the intersection of the halfplane $\varphi > 0$ and the halfplane $W < 0$. One can easily note that the continuous branch $\psi = \psi_1(\varphi)$ of the isocline of zero (the curve $P_1(\varphi, \psi) = 0$) of equation (2.8) belongs to the sector I.

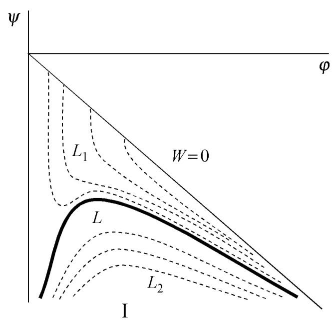

Fig. 1

This branch as $\varphi \to 0$ has the vertical asymptote $\varphi = 0$, and as $\varphi \to \infty$ an inclined asymptote $\psi = k_1\varphi$, where the angular coefficient obeys the relation

$$k_1 = -\frac{\beta\delta}{2}(2n^2 + 7n + 4 + (4n^4 + 16n^3 + 21n^2 + 8n)^{1/2});$$

in addition, $\psi_1 \sim -(\alpha\varphi^{-n-1})^{1/n}$ as $\varphi \to 0$. Using this fact, one can prove that in the sector I two families of integral curves of equation (2.8) are present. The curves of the family L_1 start (see

Fig. 1) at the points of the straight line A: $W = 0$. As $\varphi \to \infty$, they have the straight line A as an inclined asymptote; moreover, the asymptotics takes place

$$W(\varphi,\psi) \sim B_1\varphi^{-n\alpha}. \tag{2.9}$$

In relation (2.9) the coefficient $B_1 = B_1(\varphi_0) < 0$ ($\varphi_0 > 0$) is uniquely defined by the point (φ_0, ψ_0), at which the integral curve of the family L_1 starts. These curves possess either no extremal points, or two extremal points (one maximum and one minimum) situated on the zero isocline ψ_1. The integral curves of the family L_2 possess the vertical asymptote $\varphi = 0$ and the inclined asymptote, which is the straight line A; moreover, for them the asymptotics are valid

$$\psi \sim A_1\varphi^{-1/(n\beta)} \quad (\varphi \to 0), \quad W(\varphi,\psi) \sim c_1(A_1)\varphi^{-n\alpha} \quad (\varphi \to \infty). \tag{2.10}$$

Here the coefficients $A_1 < 0$ and $c_1 = c_1(A_1) < 0$ are defined by the integral curve of the family L_2, we always have $B_1(\varphi_0) > c_1(A_1)$. The curves of the family L_2 intersect the zero isocline ψ_1 at the point of maximum. The families L_1 and L_2 are divided by a curve L lying in the sector I, which is an integral curve with the asymptotic

$$\psi \sim -\left(\frac{\beta}{\alpha}\varphi^{n+1}\right)^{-1/n} \qquad (\varphi \to 0), \tag{2.11}$$

$$W(\varphi,\psi) \sim B\varphi^{-n\alpha} \qquad (\varphi \to \infty). \tag{2.12}$$

The coefficient B in relation (2.12) can be determined only numerically. Note that the inequalities $B_1(\varphi_0) > B > c_1(A_1)$ hold always (i. e., for all $\varphi_0 > 0$ and $A_1 < 0$).

By virtue of relations (2.6), (2.7) the one-parametric families L_1 and L_2 generate two-parametric families of nonnegative decreasing for $\xi > 0$ solutions of equation (2.3); the curve L, in turn, generates a one-parametric family of solutions (the parameter of the family is a point of the front ξ_0 of the solution $V(\xi)$). By means of asymptotic relations (2.9), (2.10) one can easily verify that the solutions generated in the halfplane $\xi > 0$ by integral curves of the families L_1 and L_2 cannot satisfy the boundary condition (2.4) and the condition of the continuity of the flow q for $\xi > 0$.

On the contrary, from relation (2.11) it follows

$$\varphi \sim c_n\left(\frac{\xi_0 - \xi}{\xi_0}\right)^{n\delta} \quad (\xi \to \xi_0 - 0, \ \ \xi_0 > 0), \quad c_n = (n\delta)^{-n\delta}(\alpha/\beta)^{\delta};$$

whence for nonnegative decreasing for $\xi > 0$ solutions V_1 of equation (2.3), which are generated by the curve L, to the left from the point of the front ξ_0, which yet remains arbitrary, we obtain the asymptotic representations

$$V_1 \sim c_n\xi_0^{\delta}(\xi_0 - \xi)^{n\delta} \quad (\xi \to \xi_0 - 0). \tag{2.13}$$

By means of relations (2.7) and (2.11) we determine the asymptotic of the derivative

$$V_1' \sim -n\delta c_n\xi_0^{\delta}(\xi_0 - \xi)^{-\gamma} \quad (\xi \to \xi_0 - 0). \tag{2.14}$$

From (2.13), (2.14) one can easily obtain that at the point of front ξ_0 of the solution V_1 the flow vanishes $q = 0$; therefore, by continuing the solution V_1, constructed on the segment $0 \le \xi \le \xi_0$, by means of the relation $V_1(\xi) = 0$ for $\xi > \xi_0$, we obtain a nonnegative nonincreasing solution of equation (2.3), defined on the half-axis $\xi \ge 0$, satisfying both the boundary condition (2.4) and the condition of continuity of the flow for $\xi > 0$.

Now let us use the asymptotic (2.12). We find that $\varphi \sim V_{10}\xi^{-\gamma}$ ($\xi \to +0$); whence by virtue of (2.6), (2.7) we obtain

$$V_1(0) = V_{10}, \quad V_1'(0) = BV_{10}^{-n\alpha}. \tag{2.15}$$

The constant V_{10} in (2.15) is defined in one-to-one way by the point of front ξ_0 of the solution $V_1(\xi)$.

In order to continue the solution constructed V_1 to the halfplane $\xi < 0$, as in (2.6) we put

$$V(\xi) = (-\xi)^\gamma \varphi(\xi), \quad \psi = \xi\varphi'_\xi, \quad \xi < 0; \tag{2.16}$$

whence we find

$$V'_\xi = -(-\xi)^{n\delta} W(\varphi, \psi). \tag{2.17}$$

After substitution of relations (2.16), (2.17) into equation (2.3) we obtain

$$\frac{d\psi}{d\varphi} = -\frac{P_2(\varphi,\psi)}{Q_2(\varphi,\psi)}, \tag{2.18}$$

where

$$P_2 = \frac{\alpha}{\beta} + \varphi^{n+1}W^{n-2}[\gamma n\varphi\psi + \delta(3n+2)\varphi W + \tfrac{1}{\beta}\psi W], \quad Q_2 = n\varphi^{n+2}\psi W^{n-2}.$$

Nonnegative decreasing in the halfplane $\xi < 0$ solutions $V(\xi)$ of equation (2.3) by virtue of relations (2.16), (2.17) are generated by integral curves of equation (2.18), which belong to sector II of the plane φ, ψ, which is the intersection of the halfplanes $\varphi > 0$ and $W > 0$ (see Fig. 2).

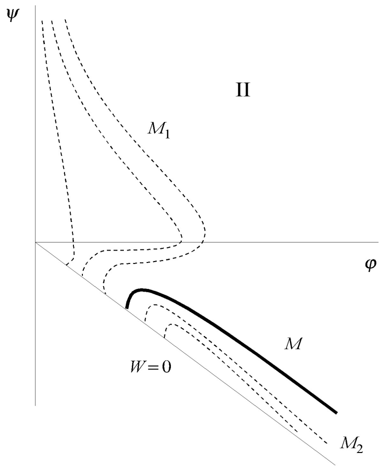

Fig. 2

We have to consider separately three cases: 1) $n > 2$; 2) $n = 2$; 3) $1 < n < 2$. In all these cases the zero isocline of equation (2.19) has in the sector II the branch ψ_2 with inclined asymptote $\psi = k_2\varphi$, where

$$k_2 = -\frac{\beta\delta}{2}(2n^2 + 7n + 4 - (4n^4 + 16n^3 + 21n^2 + 8n)^{1/2}).$$

In case 1), the isocline of zero has in the sector II also the second branch ψ_3 with the inclined asymptote $W = 0$; moreover, the asymptotic is valid

$$\psi_3 \sim -\gamma\varphi + c\varphi^{-(n+3)\mu}, \quad c = \left(\frac{\alpha^3}{n\beta\delta^2}\right)^\mu \quad (\varphi \to \infty).$$

The branches ψ_2 and ψ_3 cannot intersect the boundary of the sector II and the axis φ and are connected to each other at a certain interior point of the sector II. In case 2), the branch ψ_3 disappears, while the branch ψ_2 intersects the boundary of the sector II at the point $((50/27)^{1/5}, -3(50/27)^{1/5}/5)$, belonging to the straight line $W = 0$. In case 3), by rewriting equation (2.18) in the equivalent form $\psi'_\varphi = -P_3/Q_3$ ($P_3 = P_2W^{2-n}$, $Q_3 = n\varphi^{n+2}\psi$), we again obtain that the branch ψ_3 vanishes while the branch ψ_2 intersects the straight line $W = 0$ at the origin. Using these results, investigating the behavior of the integral curves we can prove that in every of these three cases in the sector II the two families of integral curves of equation (2.18) exist (see Fig. 2). The curves of both the families M_1 and M_2 start at the points (φ_0, ψ_0) of the straight line $W = 0$, but the curves of the family M_1

possess the vertical asymptote $\varphi = 0$, while the curves of the family M_2 the inclined one $W = 0$. The curves of the family M_1 in case 1) either do not intersect any of the branches of the isocline of zero, or intersect both the branches ψ_2 and ψ_3 (the branch ψ_2 is intersected by the integral curve at the point of minimum, the branch ψ_3 at the point of maximum). The curves of the family M_2 intersect only the branch ψ_3, and then, decreasing as $\varphi \to \infty$, approach the asymptote $W = 0$. In cases 2) and 3) the curves of the family M_2 do not intersect the branch ψ_2 of the isocline of zero. The curves of the family M_1 in case 2) can either intersect or not the branch ψ_2, in case 3) all the curves of the family M_1 intersect the branch ψ_2. The families M_1 and M_2 are divided by the integral curve M which goes out from a certain point $(\widetilde{\varphi}_0, \widetilde{\psi}_0)$ of the straight line $W = 0$ and goes in the sector II to a singular point on ∞ in the direction with the angular coefficient

$$k_3 = -(3n+2)\alpha\delta/2, \quad k_2 > k_3 > -\gamma. \tag{2.19}$$

For the curves of the family M_1 we have $\varphi_0 < \widetilde{\varphi}_0$, for the curves of the family M_2 — $\varphi_0 > \widetilde{\varphi}_0$. In all these three cases the asymptotic takes place for the curves of the family M_1

$$\psi \sim A_2 \varphi^{-1/n\beta} \ \ (\varphi \to 0), \tag{2.20}$$

which is analogous to the first of formulas (2.10). For the curves of the family M_2 the asymptotic is valid

$$W(\varphi, \psi) \sim B_2 \varphi^{-n\alpha} \ \ (\varphi \to 0), \tag{2.21}$$

which is similar to the second relation in (2.10). The coefficients $A_2 = A_2(\varphi_0) > 0$ and $B_2 = B_2(\varphi_0) > 0$ in formulas (2.20), (2.21) now are uniquely determined by the abscissa φ_0 of the point of the straight line $W = 0$, which is left by the integral curve; moreover, as φ_0 increases, the coefficient $A_2(\varphi_0)$ also increases, the coefficient $B_2(\varphi_0)$ continuously decreases from ∞ to 0 within the growth of φ_0 from $\widetilde{\varphi}_0$ to ∞.

Let us investigate the behavior of the integral curves of the families M_1 and M_2 near the exit point $(\varphi_0, -\gamma\varphi_0)$, which belongs to the straight line $W = 0$, i. e., as $\varphi \to \varphi_0$, $\psi \to \psi_0 = -\gamma\varphi_0$. We obtain

$$\psi - \psi_0 = d_n(\varphi - \varphi_0)^\lambda - \gamma(\varphi - \varphi_0) + O((\varphi - \varphi_0)^{n\lambda}) \ \ (\varphi \to \varphi_0 - 0), \tag{2.22}$$
$$d_n = \left(\frac{\alpha^2}{n\beta\delta\lambda\varphi_0^{n+3}}\right)^\lambda.$$

Now by means of relations (2.16), (2.17), (2.19) and asymptotic representations (2.20)–(2.22) one can easily study the behavior of the two-parametric families N_1 and N_2 of nonnegative monotonely decreasing solutions of equation (2.3), which are generated by the respective one-parametric families M_1 and M_2, and also the behavior of the one-parametric family N of nonnegative monotonely decreasing solutions generated by the integral curve M. It turns out that all the solutions of the family N vanish for $\xi = 0$, while all the solutions of the family N_1 intersect the negative half-axis $\xi < 0$ (every at respective point). Consequently, the solutions which belong to these families cannot be smoothly united at the point $\xi = 0$ with the solution V_1. Therefore, only solutions of the family N_2 are of interest. These solutions are defined for $0 > \xi > \xi_1$ (ξ_1 and φ_0 are two parameters defining the solution, the parameter ξ_1 is chosen so that $V_2'(\xi_1) = 0$, the possibility of such a choice is stipulated by the validity of asymptotic (2.22)). After rather cumbersome calculation which is omitted here, with the help of asymptotic (2.22) we obtain for the solutions of the family N_2 as

$\xi \to \xi_1 + 0$

$$V_2 = (-\xi_1)^\gamma \varphi_0 \Big[1 - \frac{\delta d_n}{n\alpha\lambda(-\psi_0)^{\lambda/\mu}} \Big(\frac{\xi_1 - \xi}{\xi_1}\Big)^{n\lambda} + O((\xi - \xi_1)^w)\Big], \quad w = \min\{2, \lambda/\alpha\},$$
$$V_2 = (-\xi_1)^{3/5} \varphi_0 \Big[1 + \frac{3}{5}\Big(\frac{d_2}{2} - \frac{8}{5}\Big)\Big(\frac{\xi_1 - \xi}{\xi_1}\Big)^2 + O((\xi - \xi_1)^3)\Big], \quad n = 2, \tag{2.23}$$
$$V_2 = (-\xi_1)^\gamma \varphi_0 [1 + o((\xi - \xi_1)^2)], \quad 1 < n < 2.$$

From (2.17) by means of (2.22) as $\xi \to \xi_1 + 0$ for all $n > 1$ we find

$$V_2' = -(-\xi_1)^{-n\delta} d_n (\gamma\varphi_0)^\lambda \Big(\frac{\xi_1 - \xi}{\xi_1}\Big)^\lambda + O((\xi - \xi_1)^{n\lambda}). \tag{2.24}$$

On the other hand, with the help of asymptotic (2.21) one can easily verify that for the solutions of the family N_2 as $\xi \to -0$ the finite $\lim V_2(\xi) = V_{20}$ exists depending on both ξ_1 and φ_0; besides, $V_2'(0) = -B_2 V_{20}^{-n\alpha}$, $B_2 = B_2(\varphi_0)$.

Now, having assumed that $\xi_1 = -\varphi_0^{1/\gamma}$, by virtue of relations (2.23) and (2.24) we obtain the solution of the family N_2, which satisfies the conditions $V_2(\xi_1) = 1$ and $V_2'(\xi_1) = 0$. Let us continue it to the values $\xi < \xi_1$ by assuming $V_2(\xi) = 1$. As a result, in the halfplane $\xi < 0$ we obtain the solution $V_2(\xi)$ which satisfies both the boundary condition (2.5) and the condition of the continuity of flow for $\xi < 0$. This solution depends on the parameter φ_0, which we choose so that

$$B_2(\varphi_0) = -B. \tag{2.25}$$

Recall that $B_2(\varphi_0)$ decreases continuously from ∞ to 0 with the growth of φ_0 from $\widetilde{\varphi}_0$ to ∞, and $B < 0$; whence it follows that the unique solution φ_0 of equation (2.25) exists.

Finally, we subordinate the choice of a point of front $\xi_0 > 0$ to the requirement

$$V_{20} = V_{10}. \tag{2.26}$$

The existence of the unique solution of equation (2.26) follows from a group property of solutions of equation (2.3), which is easily verifiable: Together with the solution $V(\xi)$ the function $\zeta^{-\gamma} V(\zeta\xi)$ is also a solution, where $\zeta > 0$ is arbitrary. Indeed, let us consider the solution $\widetilde{V}(\xi)$ of equation (2.3), which is generated by the curve L, with the point of front $\xi_0 = 1$ (it is uniquely defined) and put

$$V_1(\xi) = \zeta^{-\gamma} \widetilde{V}(\zeta\xi). \tag{2.27}$$

By means of equation (2.26) we easily find that for $\zeta = (V_{20}/\widetilde{V}(0))^{-1/\gamma}$ the solution V_1 of equation (2.3) with the point of front $\xi_0 = 1/\zeta$ and the zero flow q_1 on the front, defined by equality (2.27) satisfies both relation (2.26) and the equality

$$V_1'(0) = V_2'(0), \tag{2.28}$$

i.e., it is the desired solution. By putting

$$V(\xi) = \begin{cases} V_1(\xi), & \xi \geq 0; \\ V_2(\xi), & \xi < 0, \end{cases}$$

we obtain a solution of equation (2.3), defined on the whole axis and satisfying both the boundary conditions (2.4), (2.5) and the condition of continuity of the flow q_1, which follows from (2.15), (2.25), (2.26), (2.28).

It seems to be convenient to formulate the above algorithm for solving the problem on decay of gap and the results obtained in the following form.

Theorem. *A unique nondecreasing and continuous together with the flow* q (1.2), *nonnegative automodelling solution* $l(t, x) = V(\xi)$, $\xi = xt^{-\alpha}$ *exists for the problem of decay of gap* (1.1), (1.3).

This solution $V(\xi)$ is equal to 1 for $\xi \le \xi_1 = -\varphi_0^{-1/\gamma}$, where $\varphi_0 > 0$ is the unique solution of equation (2.25), in which $B < 0$ is the coefficient in asymptotic (2.12) and $B_2(\varphi_0) > 0$ is the coefficient in asymptotic (2.21). For $\xi_1 < \xi < 0$, the solution $V(\xi)$ coincides with the solution $V_2(\xi)$ of the ordinary differential equation (2.3) for which with $\xi \to \xi_1 + 0$ the asymptotic representations (2.23), (2.24) are valid; thus $V_2(\xi_1) = 1$, $V_2'(\xi_1) = 0$. For $0 \le \xi \le \zeta^{-1}$, the solution $V(\xi)$ is equal to $\zeta^{-\gamma}\widetilde{V}(\zeta\xi)$; here $\widetilde{V}(\xi)$ is a solution of equation (2.3) with the point of front $\xi_0 = 1$ and asymptotic (2.13), (2.14) for $\xi \to 1 - 0$ (it is definable in the unique way), while $\zeta = (V_2(0)/\widetilde{V}(0))^{-1/\gamma}$. Finally, $V(\xi) = 0$ for $\xi > \zeta^{-1}$.

3. Let us apply the results obtained in Item 2 to the construction of an automodelling discontinuous nonnegative nonincreasing solutions of the modified problem (1.1), (1.3). To this end, first of all, we introduce the necessary condition which must be fulfilled for the discontinuous solution l on the line of gap $x = x(t)$. We denote

$$l^+ = \lim_{x \to x(t)+0} l(t,x), \quad l^- = \lim_{x \to x(t)-0} l(t,x), \quad [l] = l^+ - l^-,$$
$$q^+ = \lim_{x \to x(t)+0} q, \qquad q^- = \lim_{x \to x(t)-0} q, \qquad [q] = q^+ - q^-,$$

$x_0(t)$ is a point of the front of the solution l ($l(t, x_0(t)) = 0$, $q|_{x=x_0(t)} = 0$), $x_1(t)$ is the point of union of the curvilinear part of the solution $l(t,x)$ with the straight line $l = 1$. From the law of conservation of the mass we obtain the equality

$$\int_{x_1(t)}^{x(t)-0} l(t,x)dx + \int_{x(t)+0}^{x_0(t)} l(t,x)dx = -x_1(t). \tag{3.1}$$

Differentiating identity (3.1) with respect to t, we obtain

$$x'(l^- - l^+) + \int_{x_1(t)}^{x(t)-0} l_t dx + \int_{x(t)+0}^{x_0(t)} l_t dx = 0. \tag{3.2}$$

With regard for equation (1.1) from (3.2) the condition follows

$$x'(t)[l] = -[q], \tag{3.3}$$

which must be fulfilled on the line of gap $x = x(t)$.

In the plane x, t we consider a domain D bounded from the below by the half-line $R_1 = \{(x,t) : x < 0,\ t = 0\}$, and from the right by the curve

$$R_2 = \{(x,t) : x = \xi_0 t^\alpha,\ \xi_0 > 0 \text{ is given}\}. \tag{3.4}$$

In the domain D we construct the discontinuous solution l of equation (1.1), which satisfies the following conditions on the boundary of the domain

$$l(t,x) = 1, \quad (x,t) \in R_1, \tag{3.5}$$
$$l(t,x) = 0, \quad q = 0, \quad (x,t) \in R_2. \tag{3.6}$$

In addition, on the line of gap

$$R_3 = \{(x,t) : x = \xi_2 t^\alpha,\ \xi_2 < 0 \text{ is given}\} \tag{3.7}$$

in correspondence with (3.3) we require the fulfillment of the relation

$$\alpha\xi_2 t^{\alpha-1}[l] = [\widetilde{q}], \quad \widetilde{q} = -q = \beta l^{n+2}(-l_x)^n. \tag{3.8}$$

We will consider the generalized solution of problem (1.1), (3.5)–(3.8), admitting, as usual (see [5]), that the sought-for solution can possess not only a first order discontinuity on the curve R_3, but also can fail to possess all derivatives participating in equation (1.1) also on some other curves in the domain D.

Remark. Let us note the presence of the gap in the boundary data of the problem formulated above. Indeed, from (3.5) we find $\lim l = 1$ ($x \to -0$, $(x,t) \in R_1$), and from (3.6) we have $\lim l = 0$ ($x \to +0$, $(x,t) \in R_2$).

We again will construct an automodelling nonnegative nonincreasing (at the continuity points) solution of problem (1.1), (3.5)–(3.8) (to this objective the choice of the curves R_2 and R_3 in the form (3.4) and (3.7), respectively, corresponds), assuming $l(t,x) = V(\xi)$, $\xi = xt^{-\alpha}$. From (1.1), (3.5)–(3.8) it follows that the function $V(\xi)$ must satisfy equation (2.3) for $\xi < \xi_0$ everywhere except for the point of gap ξ_2, where by virtue of (3.7), (3.8) the relation must be fulfilled

$$\alpha\xi_2[V] = [q_1], \quad q_1 = \beta V^{n+2}(-V_\xi)^n \tag{3.9}$$

(in some points of the half-axis $\xi < \xi_0$ the function V may fail to possess all derivatives which participate in equation (2.3)). The boundary conditions (3.5), (3.6) give us

$$V = 1 \qquad (\xi = -\infty), \tag{3.10}$$

$$V = q_1 = 0 \quad (\xi = \xi_0). \tag{3.11}$$

Finally, the function V must be nonnegative and nonincreasing.

Let us pass to the construction of the desired solution $V(\xi)$ of equation (2.3), which satisfies conditions (3.9)–(3.11). For $0 \le \xi \le \xi_0$, we set $V(\xi) = V_1(\xi)$, where V_1 is the solution of equation (2.3), constructed in Item 2, satisfying condition (3.11). It is uniquely defined since the point of front $\xi_0 > 0$ is given, it is nonnegative and does not increase. Thus, the values of $V_1(0) > 0$ and $V_1'(0) < 0$ are also defined. Now, for $\xi < 0$, one can construct the unique nonnegative nonincreasing solution $V_2(\xi)$ which satisfies the conditions $V_2(0) = V_1(0)$, $V_2'(0) = V_1'(0)$. This solution belongs to the family N_2 and the method for its construction is given in Item 2.

Along with the solution V_2 the values $V_2^+ = \lim V_2$, $q_1^+ = \lim q_1$, $\xi \to \xi_2 + 0$, turn to be defined; consequently, the value $m^+ = \alpha\xi_2 V_2^+ - q_1^+$ is also defined (because $\xi_2 < 0$, $V_2^+ > 0$, $q_1^+ > 0$). We set $V(\xi) = V_2(\xi)$, $\xi_2 < \xi < 0$.

For $\xi < \xi_2$, we construct the solution $V_3(\xi, \xi_1, \varphi_0)$ of equation (2.3), which belongs to the family N_1 if $0 < \varphi_0 < \widetilde{\varphi}_0$, to family N if $\varphi_0 = \widetilde{\varphi}_0$, to family N_2 in the case where $\varphi_0 > \widetilde{\varphi}_0$ (see Item 2). We will have $V_3 = (-\xi_1)^\gamma \varphi_0$, $V_3' = 0$ for $\xi = \xi_1$. Having put $\xi_1 = -\varphi_0^{-1/\gamma} < 0$, we obtain $V_3 = 1$, $V_3' = 0$ for $\xi = \xi_1$. Now it turns out that the values $V_3^-(\varphi_0) = \lim\limits_{\xi \to \xi_2 - 0} V_3(\xi, \xi_1, \varphi_0) > 0$, $q_1^- = \lim\limits_{\xi \to \xi_2 - 0} \beta V_3^{n+2}(-V_3')^n > 0$, are defined and along with them also the value $m^-(\varphi_0) = \alpha\xi_2 V_3^- - q_1^- < 0$. Relation (3.9) is equivalent to the transcendent equation

$$m^-(\varphi_0) = m^+ \tag{3.12}$$

with respect to $\varphi_0 > 0$. Having solved equation (3.12), we determine the function $V_3(\xi, \xi_1, \varphi_0)$ ($\xi_1 = -\varphi_0^{1/\gamma}$). Put $V = V_3$ for $\xi_1 < \xi < \xi_2$, $V = 1$ for $\xi < \xi_1$; then the desired solution $V(\xi)$ is obtained.

To prove the solvability of equation (3.12) we consider the behavior of the solution V_3 and its derivative V_3' when φ_0 varies from 0 to ∞. If $\varphi_0 < \widetilde{\varphi}_0$, then the curve V_3 belongs to the family N_1 and for $\varphi_0 \to 0$ we will have $\varphi(\xi) \to 0$ uniformly with respect to ξ; by virtue of (2.20), we have $W(\varphi, \psi) \sim A_2(\varphi_0)\varphi^{-1/n\beta}$. Hence and from (2.16), (2.17) we obtain with $\varphi_0 \to 0$

$$V_3^- \to 0, \quad q_1^- \sim \beta(-\xi_2)^{n+1+\delta} A_2^n(\varphi_0) \to 0,$$

because $A_2(\varphi_0) \to 0$ for $\varphi_0 \to 0$. Consequently, $m^-(\varphi_0) \to 0$ for $\varphi_0 \to 0$. For $\varphi_0 > \widetilde{\varphi}_0$, the curve V_3 belongs to the family N_2 and with $\varphi_0 \to \infty$ we will have $\xi_1 \to -0$. Hence we determine $V_3^- = 1$ (because $\xi_2 < \xi_1$ if φ_0 is sufficiently large), $q_1^- = 0$ ($V_3'(\xi_2 - 0) = 0$, $\varphi_0 \gg 1$). Thus, we have $m^- = \alpha\xi_2 < 0$ if $\varphi_0 \gg 1$. Thus, the value $m^-(\varphi_0)$, as φ_0 changes from 0 to ∞, takes at least all negative values between zero and $\alpha\xi_2$ (one can easily see that $m^-(\varphi_0)$ is a continuous function of φ_0). Consequently, the fulfillment of the inequality $m^+ > \alpha\xi_2$ will be a sufficient condition for the solvability of equation (3.12) and, at the same time, a sufficient condition of the existence of an automodelling nonnegative and nonincreasing solution $l = V(\xi)$ of the mixed problem (1.1), (3.5)–(3.8) in unbounded domain D; this inequality can be treated as a certain constraint upon the choice of the parameter ξ_0 (or ξ_2). Let us note that, in addition to the gap on the curve R_3, the constructed solution $l(t, x)$ admits also a gap of the second derivative with respect to x on the curve $x = \xi_1 t^\alpha$, $t > 0$.

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