ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 11, pp. 27–32. © Allerton Press, Inc., 2011.
Original Russian Text © E.Yu. Lerner, 2011, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2011, No. 11, pp. 34–40.

Relationship Between Matching and Assignment Problems

E. Yu. Lerner[1*]

[1]*Kazan (Volga Region) Federal University, ul. Kremlyovskaya 18, Kazan, 420008 Russia*

Received September 17, 2010; in final form, November 9, 2010

Abstract—Let $(R_{ik})^n_{i,k=1}$ and $(J_{ik})^n_{i,k=1}$ be preference matrices in the stable matching problem and let $(J_{ik})^n_{i,k=1}$ be the measure of the mutual antipathy in the assignment problem. In this paper we describe all functions f such that if $H_{i,k} = f(R_{ik}, J_{ik})$, then for any matrices R and J solution sets in stable matching and assignment problems (partly) coincide. Thus we answer the question about the relationship between these problems stated by D. Knuth. The obtained results are analogous to the Arrow theorem, and the proof techniques are close to those used in the group choice theory.

DOI: 10.3103/S1066369X11110041

Keywords and phrases: *stable matching, assignment problem, Knuth problems, preference matrix, group choice, Arrow theorem.*

1. Introduction. The stable matching problem stated in 1962 by D. Gale and L. S. Shapley [1] was studied by many authors in various branches of mathematics and in various applications (e.g., [2, 3] and references therein). In the theory of algorithms the minimal weighted matching problem (the assignment problem) [4] is known better. One can also treat it as the problem on the minimization of the total antipathy. Unlike the assignment problem which is considered in nearly all books devoted to the theory of algorithms on graphs, the stable matching problem is studied in few works in Russian ([5]; [6], pp. 43–63); they are mainly intended for experts in mathematical economics (the Moscow Center for continuous mathematical education is going to publish the Russian translation of the book [3] soon).

The establishment of the interconnection between the stable matching and assignment problems is mentioned (among other questions) in the book [3] (P. 64) by D. Knuth. The difficulty is that in the stable matching problem the antipathy of an ith man to a kth woman, as well as the antipathy of the kth woman to the ith man, are given in the initial data as separate ranks R_{ik} and J_{ik}, respectively,[1)] while in the assignment problem one value H_{ik} represents a measure of the mutual antipathy between the ith man and the kth woman. In this paper we describe all nontrivial functions f such that if $H_{ik} = f(R_{ik}, J_{ik})$, then for *any* matrices R and J all stable matchings provide the optimum in the corresponding assignment problem. It appears that if the number of men is not less than three, then all such functions f satisfy the following principle: “either the best choice or any one”, because accurate to a constant term they coincide with a function f in the form

$$f(x,y) = \begin{cases} 0, & \text{if } x = y = 1 \quad \text{(there is no mutual antipathy,} \\ & \qquad\qquad\quad \text{if both spouses love each other);} \\ c, & \text{where } c \geq 0, \quad \text{otherwise (one and the same level of unhappiness, if one of} \\ & \qquad\qquad\quad \text{spouses (or both of them) is not in love with another one).} \end{cases} \tag{1}$$

This non-diversity of the class of the desired functions has something in common with the Arrow theorem [7] which is well-known in the group choice theory. According to this theorem, if the number of parties is not less than three, then there exists only one electoral system satisfying some natural list of axioms.

*E-mail: eduard.lerner@gmail.com.

1)Following [1, 3], we use the matrimonial terminology; we denote matrices that characterize preferences of men and women by initials of well-known Shakespeare characters.