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Iterated Boolean Functions in the Elementary Basis

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Abstract—We establish a new criterion for a Boolean function to be read-once over the basis of conjunction, disjunction, and negation. We prove that each Boolean function is either read-once or has a set of four or six input vectors such that values of this function on these vectors show that it is iterated. We use this criterion to deduce an alternative proof of the known Stetsenko criterion.

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A Boolean function $f(x_1,\dots,x_n)$ which is (is not) representable by a repetition-free formula in the *elementary* basis $\{\&,\vee,\neg\}$ is called *read-once* (*iterated*). Constants 0 and 1 are said to be read-once functions.

Recall that a Boolean function $f(x_1,\dots,x_n)$ is called *monotone* (*antimonotone*) *in some variable* x_i if for any set of values of the rest variables $\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n$ the following correlation is valid:

$$f(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n)\le f(\alpha_1,\dots,\alpha_{i-1},1,\alpha_{i+1},\dots,\alpha_n),$$
$$(f(\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_n)\ge f(\alpha_1,\dots,\alpha_{i-1},1,\alpha_{i+1},\dots,\alpha_n)).$$

A function which is monotone or antimonotone in all variables is called *polarizable*.

Proposition 1. *Any read-once function is representable as a tree such that alternating conjunctions and disjunctions of arbitrary arity are realized at its inner vertices, and identical functions and negations of pairwise distinct variables are realized at its leaves.*

Proposition 2. *The set of read-once functions is closed with respect to a substitution of constants.*

Proposition 3. *Any nonpolarizable function is iterated.*

The replacement of a variable or a function itself with its negation and the permutation of variables are called *transformations of generalized homotypicity*. Functions which are obtained from each other by a finite number of transformations of generalized homotypicity are called *generalizedly homotypic*.

Proposition 4. *Generalizedly homotypic functions are, at the same time, read-once.*

An essential variable x_i of a function $g(x_1,\dots,x_m)$ is called *labeled* if both functions $g(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_m)$ and $g(x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_m)$ are not identical constants and essentially depend on all essential variables g except x_i.

We say that a function has a *prohibited quadruple* if its definition domain contains two pairs of collections neighboring in one and the same variable, on which this function coincides with this variable and its negation, respectively. Note that a function without prohibited quadruples is polarizable. We say that a function $f(x_1,\dots,x_n)$ has a *prohibited sextuple* if its definition domain or the definition domain

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