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Abelian Equations and Rank Problems for Planar Webs

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Abstract—We find an invariant characterization of planar webs of maximum rank. For 4-webs, we prove that a planar 4-web is of maximum rank three if and only if it is linearizable and its curvature vanishes. This result leads to the direct web-theoretical proof of the Poincaré theorem: A planar 4-web of maximum rank is linearizable. We also find an invariant intrinsic characterization of planar 4-webs of rank two and one and prove that in general such webs are not linearizable. This solves the Blaschke problem "to find invariant conditions for a planar 4-web to be of rank 1 or 2 or 3." Finally, we find invariant characterization of planar 5-webs of maximum rank and prove than in general such webs are not linearizable.

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1. INTRODUCTION

Bol in [6] (see also [4] and [3]) proved that the rank of a planar d-web does not exceed $(d-1)(d-2)/2$. Chern in [7] posed the problem: "Determine all d-webs of curves in the plane having maximum rank $(d-1)(d-2)/2$, $d \geqq 5$."

In the current paper, we find an invariant characterization of planar d-webs of maximum rank and provide a detailed description for the cases $d = 4, 5$. This is the first step for solution of Chern's problem formulated above.

For 4-webs, it is well known that the geometry of a 4-web is determined by the curvature K of one of its 3-subwebs, the basic invariant a and their (covariant) derivatives.

We present the characterization of 4-webs of maximum rank in two forms, an invariant analytic form: *A planar 4-web is of maximum rank if and only if the curvature K of one of its 3-subwebs and the covariant derivatives K_3 and K_4 of K are expressed in terms of the 4-web basic invariant a and the covariant derivatives of a up to the third order as indicated in formulas of Theorem* 9, and in a pure geometric form: *A planar 4-web is of maximum rank if and only if it is linearizable and its curvature vanishes* (Theorem 12). Note that the curvature of a 4-web is a weighted sum of curvatures of its four 3-subwebs.

As far as we know, these characterizations are the first intrinsic descriptions of 4-webs of maximum rank expressing maximum rank property in terms of the web invariants.

Note that Dou (see [10] and [11]) studied the rank problems for planar 4-webs. The conditions which he found were neither invariant nor effective. This was a reason that Blaschke (who was familiar with Dou's results) in his book [3] (see § 48, problem A_2) listed as open the following problem: "Find invariant conditions for a planar 4-web to be of rank 1 or 2 or 3."

Our characterizations of planar 4-webs of maximum rank indicated above along with characterizations of planar 4-webs of rank two and one give a complete solution of the Blaschke problem. The conditions we found are both invariant and effective, and we applied them to several examples.

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