Dehn functions of groups and the conjugacy problem

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Defining relations

Every $k$-generator group $G$ is a homomorphic image of $F(A) = F(a_1, \ldots, a_k)$, we have

$$G \cong F(A)/N$$

for a normal subgroup $N$ of $F(A)$. Therefore the group $G$ is defined up to isomorphism if $N$ is given. Usually $N$ is given as a normal closure of a subset of relators $R$. In other words, $N$ is the minimal normal subgroup containing the set $R$.

Examples

(1) The symmetric group $Sym(3) \cong \langle a, b \mid a^2, b^3, (ab)^2 \rangle$, where $a \mapsto (1, 2)$, $b \mapsto (1, 2, 3)$.

(2) $\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$.

(3) The surface group of genus 2: $\langle a, b, c, d \mid aba^{-1}b^{-1}cda^{-1}d^{-1} \rangle$. 
Assume that \( G = \langle A | R \rangle = \langle a_1, \ldots, a_k | r_1, \ldots, r_l \rangle \).

Let \( w = w(a_1, \ldots, a_k) \in F = F(A) \). Then \( w =_G 1 \) iff

\[
w =_F \prod_{i=1}^{t} u_i r_{j_i}^{\pm 1} u_i^{-1}, \text{ where } r_{j_i} \in R \text{ and } u_i \in F
\]

minimal number \( t = t(w) = \text{Area}(w) \)
Examples

\[ G = \langle a, b \mid aba^{-1}b^{-1} \rangle = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \langle a, b \mid ab = ba \rangle \]

(1) \[ a^2b^3a^{-2}b^{-3} =_G 1 \text{ and } Area(a^2b^3a^{-2}b^{-3}) = 6 \]
(2) $a^2 b^2 a^{-1} b^{-1} a^{-1} b^{-1} = G \ 1$ and

$Area(a^2 b^2 a^{-1} b^{-1} a^{-1} b^{-1}) = 3$
(3) \((a^2b^2a^{-2}b^{-2})^2 =_G 1\) and

\(\text{Area}((a^2b^2a^{-2}b^{-2})^2) = 8\)
Definition. A van Kampen diagram $\Delta$ over a presentation $G = \langle A \mid R \rangle$ is a finite, labeled, planar, connected and simply connected 2-complex such that

- For every edge $e$, $\text{Lab}(e) \in A^{\pm 1}$ and $\text{Lab}(e^{-1}) = \text{Lab}(e)^{-1}$;
- The boundary label of every face $\Pi$ is a word from $R^{\pm 1}$.

Lemma (van Kampen). A word $w$ in the alphabet $A^{\pm 1}$ is equal to 1 in $G = \langle A \mid R \rangle$ iff there exists a diagram $\Delta$ over $G$ with boundary label $w$. 

Example and proof

\[ G = \langle a, b \mid aba^{-1}b^{-1}, \rangle \]

\[ w =_{df} a^2ba^{-2}b^{-1} = F(a,b) \]

\[ a(aba^{-1}b^{-1})a^{-1} \times (aba^{-1}b^{-1}) \]
Dehn function of a finitely generated group $G$

$$d(n) = \max(\text{Area}(w) \mid w =_G 1 \text{ and } |w| \leq n)$$

**Equivalence.** Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we define $f \preceq g$ if for some positive integer $C$ and every $n$, we have $f(n) \leq Cg(Cn) + Cn$. We say that $f \sim g$ if both $f \preceq g$ and $g \preceq f$ hold.

Up to this equivalence, $d(n)$ does not depend on a finite presentation $\langle A \mid R \rangle$ of $G$. (A group $G = \langle A \mid R \rangle$ is called finitely presented if both $A$ and $R$ are finite sets.)
Dehn functions; a different approach

\[
G = \langle A \mid R \rangle. \text{ A word } w = w(A^{\pm 1}) = G 1 \text{ iff there is a derivation }
\]
\[
w = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_T = 1 \text{ (the empty word)},
\]

where \( w_{i-1} \rightarrow w_i \) is one of the elementary transformations:

- \( uaa^{-1}v \leftrightarrow uv, (a \in A^{\pm 1}) \)
- \( ur^{\pm 1}v \leftrightarrow uv, (r \in R) \)

**Dehn function** \( D(n) \):
\[
T = T(w) \text{ and } D(n) = \max\{T(w) \mid |w| \leq n, w = G 1\}
\]

In fact, \( D(n) \sim d(n) \).
Exercise:
The Dehn function of $\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle$ is quadratic

(1) A tip for a quadratic upper bound:

Prove that for every word $w = w(a, b)$ of length at most $n$ there is a derivation $w \to \cdots \to a^k b^l$ with $\leq n^2$ elementary transformations, and $k = l = 0$ if $w =_G 1$

(2) A tip for a quadratic lower bound:

Consider the words $w_n = a^n b^n a^{-n} b^{-n} \ (n = 1, 2, \ldots)$ and prove that $\text{Area}(w_n) = n^2$, i.e., prove that the following diagrams are minimal:
Isoperimetric function of a simply connected Riemannian manifold $M$

For a smooth simple curve $p$ in $M$, there is a 'pellicle' (or 'disk') bounded by $p$ such that $\text{Area}(D) \leq f(\text{length of } p)$

Proposition Let $G$ be a finitely generated group isometrically acting on a simply connected Riemannian manifold $M$. If the action is proper and cocompact, then $f_G \sim f_M$.

Examples. (1) $\mathbb{Z}^2$ acts on $\mathbb{R}^2$, $f_{\mathbb{R}^2}(x) = \frac{x^2}{4\pi}$, and $f_{\mathbb{Z}^2} \sim n^2$

(2) $G = \langle a, b, c, d \mid aba^{-1}b^{-1}c^{-1}d^{-1}\rangle$ acts on the standard hyperbolic plane. Therefore $G$ has linear isoperimetric (Dehn) function.

Groups with linear Dehn function are called (Gromov) hyperbolic.
**Proposition** The following properties of a finitely presented group $G$ are equivalent

(a) the Dehn function of $G$ is recursive (= computable);
(b) the Dehn function is bounded from above by a recursive function;
(c) the algorithmic word problem is decidable for $G$.

(c)$\Rightarrow$(b)
(1) For every word $w$ of length $\leq n$, one can decide whether it trivial or nontrivial in $G$.
(2) For every trivial word, one can find a presentation $w = \prod_{i=1}^{t} u_{r_{j_i} u_{j_i}}^{\pm 1} u_{i}^{-1}$ and bound $\text{area}(w)$ from above.
(3) This gives a recursive upper bound for the Dehn function.
Theorem (Birget, Olshanskii, Rips, Sapir, 2002).

The word problem of a finitely generated group $G$ has time complexity of class $\textbf{NP}$ iff $G$ is a subgroup of a finitely presented group with polynomial Dehn function.
More examples.

(1) Every finitely generated nilpotent group has at most polynomial Dehn function.

\begin{align*}
(2) \quad G &= \langle a, b \mid aba^{-1} = b^2 \rangle = \langle a, b \mid aba^{-1}b^{-2} \rangle \\
G \text{ has a faithful matrix representation: } a &\mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \ b &\mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\end{align*}
The minimal diagram for the equality

$$a^n b a^{-n} b a^n b^{-1} a^n b^{-1} = G 1$$

$$\text{Area}(a^n b a^{-n} b a^n b^{-1} a^n b^{-1}) = 2(1 + 2 + \cdots + 2^{n-1}) = 2^{n+1} - 2$$
(3) The Dehn function of the one-relator group

$$\langle a, b \mid (aba^{-1})b(aba^{-1})^{-1} = b^2 \rangle$$

asymptotically exceeds any multi-exponential function (but still recursive).

A.N. Platonov, 2004: It is equivalent to

$$\exp_2(\exp_2(\ldots(\exp_2(2)))\over \log_2 n)$$
How large is the set of Dehn functions?

(M.Bridson, 1999) First examples of Dehn functions of finitely presented groups $d(n) \sim n^\alpha$, where $\alpha$ is not integer.

The isoperimetric spectrum is the set of all real numbers $\alpha$ such that there is a finitely presented group with Dehn function equivalent to $n^\alpha$. (Does $5/2$ belong to the isoperimetric spectrum?)

Let $M$ be a Turing machine (deterministic or non-deterministic) accepting a language $L$. Then for every word $w \in L$, we have $\text{Time}(w)$ that is the length of the computation accepting $w$.

**Time function (or time complexity)**

$T(n) = \max(\text{Time}(w), \text{ where } w \in L, |w| \leq n)$
Theorem (M. Sapir, J.-C. Birget, E. Rips, 2002) Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

(a) $f(m + n) \geq f(m) + f(n)$ for any $m, n \in \mathbb{N}$ and

(b) the function $\sqrt[4]{f(n)}$ is equivalent to a time function of a Turing machine (in particular, $f(n) \geq n^4$).

Then there is a finitely presented group with Dehn function equivalent to $f(n)$. 
A real number $\alpha$ is computable with time $\leq f(m)$ if there exists a Turing machine which, given a natural number $m$, computes a binary rational approximation of $\alpha$ with an error $O(2^{-m})$, and the time of this computation $\leq f(m)$.

**Corollary (Sapir, Birget, Rips)** For a real number $\alpha \geq 4$, the function $n^\alpha$ is equivalent to the Dehn function of a finitely presented group if $\alpha$ is computable with time $\leq 2^{2^m}$. 
Examples of Dehn functions of groups.

\( n^\alpha \) for any algebraic real number \( \alpha \geq 4 \)

\( n^{\pi + \sqrt{e}} \)

\( n^k (\log n)^l \),

\( n^k (\log n)^l (\log \log n)^m \), for natural exponents \( k, l, m \) \( (k \geq 4) \)

\[ \ldots \]
Theorem (Sapir 2002, 2018) If \( n^\alpha \) is a Dehn function, then \( \alpha \) is computable in time \( \leq 2^{2^m} \).

Moreover, if \( \text{NP} = \text{P} \), then \( \alpha \) is computable in time \( \leq 2^{2^m} \).
What about the functions $n^\alpha$ for $\alpha < 4$?

If $d(n) = o(n^2)$, then $d(n) = O(n)$, that is $G$ is hyperbolic.

If $\alpha \geq 2$ and $\alpha$ is computable with time $\leq 2^{2^m}$, then there is a finitely presented group with Dehn function equivalent to $n^\alpha$.

Corollary (Sapir, Birget, Rips, Olshanskii). The isoperimetric spectrum contains 1 and all real numbers $\alpha \geq 2$ computable with time $\leq 2^{2^m}$. Provided the equality $\text{NP} = \text{P}$ is true, there are no other numbers in the isoperimetric spectrum.
The functions $n^\alpha (\log n)^\beta$, $n^\alpha (\log n)^\beta (\log \log n)^\gamma$, etc., are also Dehn functions of finitely presented groups if the exponents $\alpha, \beta, \gamma$ are computable in reasonable time.

**Corollary** If a real number $\alpha \geq 2$ is computable with time $\leq 2^{2^m}$, then there exists a closed connected Riemannian manifold $X$ such that the isoperimetric function of the universal cover $\tilde{X}$ is equivalent to $n^\alpha$. 
There exists a finitely presented group $G$ such that the Dehn function of $G$ is not bounded from above by any recursive function, but bounded by a quadratic function on an infinite set of positive integers.
Conjugacy problem in groups with quadratic Dehn function

Annular diagram for conjugacy

of two words $u$ and $v$ in $G$:

$$u = z^{-1}vz$$, i.e., $u^{-1}z^{-1}vz = 1$ in $G$

Given two words $u$ and $v$ in the generators of $G$, is there a word $z$ such that $u = z^{-1}vz$ in $G$? There is an algorithm solving this problem in a hyperbolic group $G$.

The smallest Dehn function for non-hyperbolic groups is quadratic.
Examples of finitely presented groups with quadratic Dehn function and decidable conjugacy problem

Groups acting geometrically on $CAT(0)$ spaces (Bridson, Hae-flinger, 1999, G. Kokarev, 2013).


$SL(n, \mathbb{Z}), n \geq 5$ (Grunewald, R. Sarkisyan, 1980, R. Young, 2013).
Examples of finitely presented groups with quadratic Dehn function and decidable conjugacy problem (continuation)

Thompson group $F$ (Guba and Sapir (1997), Guba (2006)).

Free-by-cyclic groups (Olshanskii and Sapir (2006), M. Bridson and D. Groves (2010))

Many metabelian (non-nilpotent) groups, some and some groups that are obtained by using the Baumslag-Remeslennikov construction (Noskov (1982), Drutu (2004), Y. de Cornulier and R. Tessera, (2010)).

High rank integral Heisenberg groups $H_n$ (N. Blackburn (1965), D. Allcock (1998), Olshanskii and Sapir (1999))

Abelian groups (obviously).
E. Rips (1994): Does every finitely presented group with quadratic Dehn function have decidable conjugacy problem?

A.O. and M.Sapir (2006) confirmed Rips’ conjecture for 'aspherical' groups, but constructed a finitely presented group with Dehn function $n^2 \log n$ and undecidable conjugacy problem.

Turing machine

Rule (command)

\[ \theta : \sigma q_1 \rightarrow \sigma q_2 b \]
S-machine

A rule (example)

\[ \theta : q_1 \rightarrow q_2 a, \quad \theta^{-1} : q_2 \rightarrow q_1 a^{-1} \]

Group generators and relations corresponding to \( \theta \).

\[ \theta^{-1}q_1\theta = q_2 a, \quad \theta^{-1}b\theta = b \]

for every tape letter \( b \).
\(\theta\)-band in diagrams

Below all vertical edges are labelled by \(\theta\)
By $\mathcal{D}$, we denote Euclidean disk. Let $T$ be a finite set of disjoint chords and $Q$ a finite set of disjoint segments inside $\mathcal{D}$. A segment $Q \in Q$ and a chord $T \in T$ may share at most one point.
We say that the pair \((T, Q)\) is a **design**.

The **length** \(\ell(Q)\) of \(Q\) is the number of the chords crossing \(Q\).

By definition, a segment \(Q_1\) is **parallel** to a segment \(Q_2\), and we write \(Q_1 \parallel Q_2\) if every chord crossing \(Q_1\) also crosses \(Q_2\).
Definition. Given $\lambda \in (0; 1)$ and an integer $n \geq 2$, the property $\mathcal{P}(\lambda, n)$ of a design says that for any $n$ different segments $Q_1, \ldots, Q_n$, there exist no subsegments $P_1, \ldots, P_n$, respectively, such that $\ell(P_i) > (1 - \lambda)\ell(Q_i)$ for every $i = 1, \ldots, n$ and $P_1 \parallel P_2 \parallel \cdots \parallel P_n$.

Lemma (A.O.) There is a constant $C = C(\lambda, n)$ such that for any design $(T, Q)$ with property $\mathcal{P}(\lambda, n)$, we have

$$\sum_{Q \in Q} \ell(Q) \leq C(\#T),$$

where $\#T$ is the number of chords in $T$. 