ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2014, Vol. 58, No. 7, pp. 1–10. © Allerton Press, Inc., 2014.
Original Russian Text © A.G. Baskakov, N.S. Kaluzhina, D.M. Polyakov, 2014, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2014, No. 7, pp. 3–14.

Slowly Varying at Infinity Operator Semigroups

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Received December 19, 2012

Abstract—In this paper we study the asymptotic behavior of bounded semigroups of linear operators acting in Banach spaces. The obtained results are closely connected with stability conditions for solutions to parabolic equations under unrestricted growth of time. Here we make no usual assumption on the existence of the mean value of the initial function.

DOI: 10.3103/S1066369X14070019

Keywords: *slowly varying functions, semigroups of linear operators, Beurling spectrum, harmonic analysis.*

1. INTRODUCTION

One of the first works devoted to the qualitative theory of parabolic equations were those by A. N. Tikhonov [1], A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov [2]. In [1] one studied the behavior as $t \to \infty$ of the solution to the first boundary-value problem for the heat conductivity equation, and in [2] one considered the behavior as $t \to \infty$ of the solution to the Cauchy problem for an almost linear second-order parabolic equation with one spatial variable. See [3] for a review of papers published before 1961. In [4–7] one studied the (pointwise and uniform) stabilization of solutions to parabolic equations as $t \to \infty$. See [8] for a review of recent related works. In [9] one considers the parabolic heat conductivity equation without the usual assumption on the existence of the mean value of the initial function. One has stated the equation under consideration as an abstract parabolic equation and used the corresponding operator semigroup. As was proved in the mentioned paper, solutions to the parabolic equation are slowly varying functions. In this paper, we extend results obtained in [9] to arbitrary bounded operator semigroups.

Let X be a complex Banach space and let $\operatorname{End} X$ denote the Banach algebra of linear bounded operators acting in X. The symbol $\mathbb{J}$ stands for one of intervals $\mathbb{R}_+ = [0, \infty)$ or $\mathbb{R} = (-\infty, \infty)$.

Denote by $C_b(\mathbb{J}, X)$ the Banach space of continuous and bounded on $\mathbb{J}$ functions, whose values belong to X, with the norm $\|x\|_\infty = \sup\limits_{t \in \mathbb{J}} \|x(t)\|$. Let symbols $C_{b,u}(\mathbb{J}, X)$ and $C_0(\mathbb{J}, X)$ stand for closed subspaces of uniformly continuous and vanishing at infinity functions from $C_b = C_b(\mathbb{J}, X)$. In $C_b(\mathbb{J}, X)$ one can define the semigroup (it is a group, if $\mathbb{J} = \mathbb{R}$) of operators $S : \mathbb{J} \to \operatorname{End} C_b$ of shift functions in the form

$$(S(t)x)(\tau) = x(\tau + t), \quad t, \tau \in \mathbb{J}.$$

A function $x \in C_b(\mathbb{J}, X)$ is said to be *slowly varying at infinity*, if

$$S(u)x - x \in C_0(\mathbb{J}, X) \text{ for any } u \in \mathbb{J}.$$

Examples of such functions are the following functions from the space $C_{b,u}(\mathbb{R}, \mathbb{C})$ (as well as their narrowings on $\mathbb{R}_+$):

1) $x_1(t) = \sin \ln(1 + t^2)$, $t \in \mathbb{R}$;

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