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TWO-METRIC SPACES

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In this paper, we construct two-dimensional two-metric manifolds. We construct two-metric functions, determine compatible connections and isometry groups. We give thermodynamic interpretation of these spaces.

1. Two-metric planes. In the widest sense, a two-metric geometry is a geometry with two distances. In such a geometry, to a pair of point there are assigned two numbers instead of one as is usual. In [1], the author gives a complete classification of two-dimensional two-metric phenomenologically symmetric geometries, i. e., geometries with the following property. There are a manifold M which is a dense submanifold in the direct product $N \times N$ of a two-dimensional manifold N by itself and a smooth nondegenerate function $f : M \to R^2$ called a *metric function*, or a *two-metric*, or a *two-metric function*. The components of f are denoted by (f^1, f^2). There is a smooth function of six variables $\Phi : R^3 \to R^2$ with components (Φ_1, Φ_2) such that $\operatorname{rang} \Phi = 2$ and, for any cortege of three arbitrary points $\langle xyz \rangle$ each pair of which belongs to M, the following relation holds:

$$\Phi(f(xy), f(xz), f(yz)) = 0.$$

There exist only two two-dimensional phenomenologically symmetric two-metric structures. These are the geometries whose metric functions have the following coordinate expressions:

$$f^1(xy) = x^1 - y^1, \qquad f^2(xy) = x^2 - y^2; \tag{1}$$

$$f^1(xy) = (x^1 - y^1)x^2, \quad f^2(xy) = (x^1 - y^1)y^2, \tag{2}$$

where (x^1, x^2) and (y^1, y^2) are, respectively, the coordinates of x and y. A space with two-metric structure will be denoted by F^2. Note that the domains of definition of the two-metric functions (1) and (2) coincide with the entire direct product $F^2 \times F^2$. As a set, F^2 coincides with the affine plane.

Consider two infinitely close points $y = (x^1, x^2)$ and $x = (x^1 + dx^1, x^2 + dx^2)$. At these points, the metric functions (1) and (2) take the following values:

$$f^1 = dx^1, \qquad f^2 = dx^2; \tag{1$'$}$$

$$f^1 = (dx^1)(dx^2 + x^2), \quad f^2 = (dx^1)x^2. \tag{2$'$}$$

Passing in (1′) and (2′) from the special coordinates (x^1, x^2) to arbitrary coordinates (y^1, y^2) which are related to (x^1, x^2) by $x^1 = \psi^1(y^1, y^2)$, $x^2 = \psi^2(y^1, y^2)$, we obtain

$$f^1 = a^1_i dy^i, \qquad f^2 = a^2_i dy^i;$$

$$f^1 = (a^1_i dy^i)(a^2_i dy^i + \xi^2), \quad f^2 = a^1_i dy^i \xi^2,$$

where $a^i_j = \partial\psi^i/\partial y^j$, $\xi^2 = \psi^2$, $i, j = 1, 2$.