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NEVANLINNA–PICK PROBLEM IN CLASS $\mathcal{S}[a, b]$

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1. Introduction

In his investigation of the power moment problem on a compact interval, M.G. Kreĭn introduced in [1] (p. 527) a certain special class of holomorphic functions $\mathcal{S}[a, b]$. This class was used to describe all solutions of an indefinite power moment problem on a compact interval. As is known (see [2], p. 121), the solution of the moment problem can be reduced to the description of the Nevanlinna functions with prescribed asymptotics along the imaginary axis. In these terms, one can assume that, in solving the moment problem on compact interval, the interpolation problem with multiple node of interpolation at infinite point for functions of the class $\mathcal{S}[a, b]$ was solved.

In this article we pose and solve the Nevanlinna–Pick problem in the class $\mathcal{S}[a, b]$ in the case where simple complex interpolation nodes are given. In addition, we consider the case of matrix-valued functions. We should note that the moment problem on compact interval in matrix statement was considered in [3].

The main results of this article are Theorems 8 and 9, where all solutions of the completely indefinite Nevanlinna–Pick problem in the class $\mathcal{S}[a, b]$ are described and criterion of solvability of the corresponding interpolation problem is established.

In the article we use the technique by V.P. Potapov for solving interpolation problems (see [4]) and some its generalizations for Nevanlinna (see [5]) and Stieltjes (see [6]–[8]) functions.

2. Statement of the Nevanlinna–Pick problem in the class $\mathcal{S}[a, b]$

Let an integer number $m \geq 1$ be given. We denote by $\mathcal{R}$ (and call it Nevanlinna) a set of $m \times m$-matrix functions $w(z)$ which are definite and holomorphic in the upper halfplane $\operatorname{Im} z > 0$ and satisfy the condition $(w(z) - w^*(z))/2i \geq 0$.

Let $[a, b]$ be a finite interval in $\mathbb{R}$. We denote by $\mathcal{S}[a, b]$ a set of $m \times m$-matrix functions which are definite and holomorphic in the halfplane $\operatorname{Im} z > 0$, are definite and continuous at the points $x \in (-\infty, a) \cup (b, +\infty)$, and satisfy the conditions

1. $\{s(z) - s^*(z)\}/\{2i\} \geq 0$, $\operatorname{Im} z > 0$,
2. $s(x) \geq 0$, $x \in (-\infty, a) \cup (b, +\infty)$.

From this definition one can see that the matrix functions $s(z) \in \mathcal{S}[a, b]$ by Schwartz symmetry principle $s(z) = s^*(\overline{z})$ can be analytically continued through $(-\infty, a) \cup (b, +\infty)$ to the lower half-plane. Therefore, in fact, they are definite and holomorphic in the whole complex plane with the cut along the interval $[a, b]$.

Let us formulate the following two theorems (see [1], p. 528).