ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2013, Vol. 57, No. 10, pp. 54–58. © Allerton Press, Inc., 2013.
Original Russian Text © M.A. Aukhadiev, S.A. Grigoryan, and E.V. Lipacheva, 2013, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2013, No. 10, pp. 55–62.

On Compact Quantum Semigroup $QS_{\rm red}$

M. A. Aukhadiev*, **S. A. Grigoryan****, **and E. V. Lipacheva*****
(Submitted by D.Kh. Mushtari)

Kazan State Power Engineering University, ul. Krasnosel'skaya 51, Kazan, 420066 Russia
Received February 22, 2013

Abstract—We study some properties of a reduced semigroup C^*-algebra of a semigroup S. For the semigroup C^*-algebra generated by the deformation of the algebra of continuous functions on a compact abelian group we obtain a structure of a compact quantum semigroup. We also consider morphisms of constructed compact quantum semigroups.

DOI: 10.3103/S1066369X1310006X

Keywords and phrases: *C^*-algebra, compact quantum semigroup, isometric representation, morphisms of quantum semigroups.*

1. Introduction. There exist two different approaches to quantization, namely, the algebraic and topological ones. The first of them (proposed by V. G. Drinfeld [1]) implies the deformation of universal enveloping algebras. The second approach (proposed later by S. L. Woronowicz [2]) is related to the theory of compact quantum groups and semigroups. In [3] one has proved that these two approaches are equivalent. Let us illustrate the quantization process in the framework of the theory of C^*-algebras.

Let P be a compact semigroup, i.e., a compact Hausdorff space with a continuous associative operation $(x, y) \to xy$. Denote by $C(P)$ the algebra of continuous functions on P. Then $C(P)$ is a commutative unital C^*-algebra which contains all topological information on the space P. Let us identify $C(P \times P)$ and $C(P) \otimes C(P)$, and define a map $\Delta : C(P) \to C(P) \otimes C(P)$ as follows:

$$\Delta(f)(x, y) = f(xy).$$

Evidently, Δ is a continuous unital $*$-homomorphism. The associativity of multiplication in P consists in the so-called *co-associativity* condition for Δ, namely,

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

Thus, all information on the compact semigroup P is contained in the pair $(C(P), \Delta)$.

Now, conversely, let $\mathcal{A}$ be some commutative unital C^*-algebra. Then by the Gelfand theorem, $\mathcal{A}$ is isomorphic to the algebra of continuous functions $C(P)$ on some compact Hausdorff space P. If on $\mathcal{A}$ an $*$-homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfying the co-associativity condition is given, then the equality $f(xy) = \Delta(f)(x, y)$ defines a semigroup structure on P.

In essence, a quantization results in the transfer from the commutative algebra $C(P)$ to the noncommutative unital C^*-algebra $\mathcal{A}$. We can treat $\mathcal{A}$ as the algebra of continuous functions on some imaginary compact geometric object, which is said to be a *quantum space*.

A unital $*$-homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfying the co-associativity condition is called the *co-product*. Analogously to the classical case, Δ turns a quantum space into a quantum semigroup. Then the algebra $\mathcal{A}$ with a (given on it) co-product is the *algebra of functions on a quantum semigroup*. The pair $(\mathcal{A}, \Delta)$ is usually called a *compact quantum semigroup* [4]. See [5] for an example of a compact quantum semigroup on a noncommutative C^*-algebra. Such objects were also studied by K. Kawamura [6].

*E-mail: m.aukhadiev@gmail.com.
**E-mail: gsuren@inbox.ru.
***E-mail: elipacheva@gmail.com.