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APPROXIMATE SOLUTION OF OPTIMAL CONTROL PROBLEM FOR A SINGULAR ELLIPTIC TYPE EQUATION WITH NONSMOOTH NONLINEARITY

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We consider an optimal control problem for a system which is described by a nonlinear elliptic type equation. No assumptions about the single-valued solvability of the corresponding boundary-value problem are made. The additional difficulty is stipulated by the fact that the state operator is not differentiable. We propose to investigate this problem using the modified penalty method with the smooth approximation of the operator. By minimization of the corresponding approximating functional we find an approximate solution of the initial optimization problem.

1. Problem definition

On the open bounded area Ω of the space $\mathbb{R}^n$ the equation

$$\Delta y + g(y) = v + f \tag{1}$$

with the homogenous boundary condition is given (the Dirichlet problem). Here v is a control, f is the known function from the space $H^{-1}(\Omega)$, g is the given function with respect to the system state y. By the Sobolev theorem, the following continuous implications hold:

$$H_0^1(\Omega) \subset L_q(\Omega), \quad H^{-1}(\Omega) \subset L_{q'}(\Omega),$$

where $1/2 - 1/n = 1/q$, $n > 2$, q is an arbitrary value, $n = 2$; $1/q + 1/q' = 1$.

Then there exists a positive constant c such that

$$\|y\|_q \le c\|y\| \quad \forall y \in H_0^1(\Omega), \quad \|y\|_* \le c\|y\|_{q'} \quad \forall y \in L_{q'}(\Omega),$$

where $\|y\|$, $\|y\|_q$ and $\|y\|_*$ are the norms of the function y in the spaces $H_0^1(\Omega)$, $L_q(\Omega)$, and $H^{-1}(\Omega)$, respectively. We assume that the function g belongs to the class

$$G = \{g \in C(\mathbb{R}) \mid |g(y)| \le a + b|y|^{q-1} \ \forall y\}, \quad a > 0, \quad b > 0.$$

Using the Krasnoselskii theorem (see [1], p. 312), we prove that the operator $g : L_q(\Omega) \to L_{q'}(\Omega)$ is continuous. We choose the control v from the convex closed subset U of the space $L_2(\Omega)$.

Under the given constraints imposed on the nonlinear term, equation (1) has no a priori estimate, that is why neither the existence of a solution of the boundary-value problem for an arbitrary control nor its uniqueness can be guaranteed on conditions that the resolvability is realized. Moreover, under certain conditions, the one-valued resolvability certainly does not hold (see, e. g., [2], p. 262). According to the general concept of the solution of extremal problems, for singular systems [2] we use the notion of an admissible pair. We consider the space $W = L_2(\Omega) \times H_0^1(\Omega)$ and the set $W_0 = \{(v, y) \in W \mid v \in U\}$.