Cosmology in Bigravity Theories

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December 6-7, 2016

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Preliminary

"bigravity" = system of massive spin 2 field (massive graviton)
+ gravity (includes massless spin 2 field = graviton)

$F(R)$ extension of bigravity, Application to Cosmology

Mainly based on

S. Nojiri and S. D. Odintsov,
"Ghost-free $F(R)$ bigravity and accelerating cosmology,"
Phys. Lett. B **716**, 377 (2012) [arXiv:1207.5106 [hep-th]].

S. Nojiri, S. D. Odintsov, and N. Shirai,
"Variety of cosmic acceleration models from massive $F(R)$ bigravity,"
JCAP **1305** (2013) 020 [arXiv:1212.2079 [hep-th]].

Massive Gravity (theory of massive spin two field)

Fierz-Pauli action (linearized or free theory), 3/4 century ago
M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," Proc. Roy. Soc. Lond. A **173** (1939) 211.

The Lagrangian of the massless spin-two field (graviton) $h_{\mu\nu}$ is given by

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\lambda h^\lambda{}_\mu \partial_\nu h^{\mu\nu} - \partial^\mu h_{\mu\nu}\partial^\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \,, \quad \left(h \equiv h^\mu{}_\mu\right) \,.$$

Massless graviton: 2 degrees of freedom (helicity),
Massive graviton: 5 degrees of freedom ($2s+1$, spin $s=2$).

The Lagrangian of the massive graviton with mass m is given by

$$\mathcal{L}_m = \mathcal{L}_0 - \frac{m^2}{2}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) \quad \text{(Fierz-Pauli action)} \,.$$

When $m = 0$, gauge symmetry (linearized general covariance)

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu ,$$

$\xi_\mu(x)$: space-time dependent gauge parameter.

The combination $h_{\mu\nu} h^{\mu\nu} - h^2$:
Fierz-Pauli tuning (not related with any symmetry)

For the combination $h_{\mu\nu} h^{\mu\nu} - (1-a) h^2$,
if $a \neq 0$, there appears ghost scalar field (later) with mass

$$m_g^2 = \frac{3-4a}{2a} m^2 , \quad \left(m_g^2 \to \infty \, \text{when } a \to 0 \right) .$$

Hamiltonian and counting of degrees of freedom:

$\frac{D(D-1)}{2} - 1$ propagating degrees of freedom in D dimensions (5 degrees of freedom for $D = 4$).

Legendre transformation only with respect to the spatial components h_{ij}.

$$\begin{aligned}
\pi_{ij} =& \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i} h_{j)0} + 2\partial_k h_{0k}\delta_{ij} \,, \\
\Rightarrow \ S =& \int d^D x \left\{ \pi_{ij}\dot{h}_{ij} - \mathcal{H} + 2h_{0i}\left(\partial_j \pi_{ij}\right) + m^2 h_{0i}^2 \right. \\
& \left. + h_{00}\left(\vec{\nabla}^2 h_{ii} - \partial_i\partial_j h_{ij} - m^2 h_{ii}\right) \right\} \,, \\
\mathcal{H} =& \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij}\partial_k h_{ij} - \partial_i h_{jk}\partial_j h_{ik} \\
& + \partial_i h_{ij}\partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj}\partial_i h_{kk} + \frac{1}{2}m^2 (h_{ij}h_{ij} - h_{ii}^2) \,.
\end{aligned}$$

$m = 0$ case: h_{0i}, h_{00}: Lagrange multipliers $\rightarrow$ constraints

$$\partial_j \pi_{ij} = 0 \,, \quad \vec{\nabla}^2 h_{ii} - \partial_i \partial_j h_{ij} = 0 \,.$$

First class constraints $\rightarrow$ gauge symmetry ($\Leftarrow$ general covariance)

For $D = 4$, h_{ij} and π_{ij} each have 6 components, respectively.
$\rightarrow$ 12 dimensional phase space.
4 constraints + 4 gauge invariances
$\rightarrow$ 4 dimensional phase space
(two polarizations (helicities) of massless graviton)

$m \neq 0$: h_{0i} are no longer Lagrange multipliers $\delta h_{0i} \;\Rightarrow\; h_{0i} = -\frac{1}{m^2}\partial_j \pi_{ij}$,

$$S = \int d^D x \left\{ \pi_{ij} \dot{h}_{ij} - \mathcal{H} + h_{00} \left(\vec{\nabla}^2 h_{ii} - \partial_i \partial_j h_{ij} - m^2 h_{ii} \right) \right\} ,$$

$$\mathcal{H} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{2}\frac{1}{D-2}\pi_{ii}^2 + \frac{1}{2}\partial_k h_{ij} \partial_k h_{ij} - \partial_i h_{jk} \partial_j h_{ik}$$

$$+ \partial_i h_{ij} \partial_j h_{kk} - \frac{1}{2}\partial_i h_{jj} \partial_i h_{kk} + \frac{1}{2} m^2 \left(h_{ij} h_{ij} - h_{ii}^2 \right) + \frac{1}{m^2} \left(\partial_j \pi_{ij} \right)^2 .$$

h_{00}: Lagrange multiplier $\rightarrow$ single constraint

$$\mathcal{C} = -\vec{\nabla}^2 h_{ii} + \partial_i \partial_j h_{ij} + m^2 h_{ii} = 0 ,$$

Secondary constraint:

$$\{H, \mathcal{C}\}_{\mathrm{PB}} = \frac{1}{D-2} m^2 \pi_{ii} + \partial_i \partial_j \pi_{ij} = 0 , \quad H = \int d^d x \, \mathcal{H} ,$$

Two second class constraints: h_{ii} and π_{ii} (trace part) can be removed.

For $D = 4$,
12 dimensional phase space $-$ 2 constraints $=$ 10 degrees of freedom
(5 polarizations of the massive graviton and their conjugate momenta).

If not the Fierz-Pauli combination, $h_{\mu\nu}h^{\mu\nu} - (1-a)h^2$, $a \neq 0$, h_{00}^2 term appears.
$\Rightarrow$ h_{ii} and π_{ii} cannot be removed.
There appear $-\pi_{ii}^2$ term (with minus sign) appears $\Rightarrow$ ghost

vDVZ(van Dam, Veltman, and Zakharov) discontinuity

H. van Dam and M. J. G. Veltman, "Massive and massless Yang-Mills and gravitational fields," Nucl. Phys. B **22** (1970) 397.

V. I. Zakharov, "Linearized gravitation theory and the graviton mass," JETP Lett. **12** (1970) 312 [Pisma Zh. Eksp. Teor. Fiz. **12** (1970) 447].

Discontinuity of $m \to 0$ limit in the free massive gravity with the Einstein gravity due to the extra degrees of freedom in the limit (Natural!).
$\Rightarrow$ the Vainstein mechanism

A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," Phys. Lett. B **39** (1972) 393.

Non-linearity screens the extra degrees of freedom (non-linearity becomes strong when m is small).

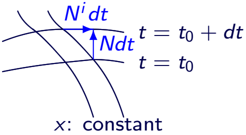
Boulware-Deser ghost

D. G. Boulware and S. Deser, "Classical General Relativity Derived from Quantum Gravity," Annals Phys. **89** (1975) 193.

In non-linear (interacting) theory, 6th degree of freedom appears as a ghost.

Non-linear massive gravity action with flat metric $\eta_{\mu\nu}$, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$

$$S = \frac{1}{2\kappa^2} \int d^D x \left[\sqrt{-g} R - \frac{1}{4} m^2 \eta^{\mu\alpha} \eta^{\nu\beta} \left(h_{\mu\nu} h_{\alpha\beta} - h_{\mu\alpha} h_{\nu\beta} \right) \right] .$$



ADM formalism (N: lapse function, N_i: shift function)

$$g_{00} = -N^2 + g^{ij} N_i N_j \, , \quad g_{0i} = N_i \, , \quad g_{ij} = g_{ij} \, ,$$

$i, j, \cdots = 1, 2, 3$, g^{ij}: inverse of the spatial metric g_{ij}.

$m = 0$ case

Einstein-Hilbert action (after partial integrations)

$$\frac{1}{2\kappa^2} \int d^D x \sqrt{g} N \left[{}^{(d)}R - K^2 + K^{ij} K_{ij} \right] ,$$

${}^{(d)}R$: curvature of spatial metric g_{ij}, K_{ij}: extrinsic curvature

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) ,$$

∇_i: covariant derivative w.r.t. the spatial metric g_{ij}.

Canonical momenta with respect to g_{ij}:

$$p^{ij} = \frac{\delta L}{\delta \dot{g}_{ij}} = \frac{1}{2\kappa^2} \sqrt{g} \left(K^{ij} - K g^{ij} \right) ,$$

Hamiltonian:

$$H = \left(\int_{\Sigma_t} d^d x \; p^{ab} \dot{g}_{ab} \right) - L = \int_{\Sigma_t} d^d x \; N\mathcal{C} + N_i \mathcal{C}^i \, .$$

$$\mathcal{C} = \sqrt{g} \left[{}^{(d)}R + K^2 - K^{ij} K_{ij} \right] \, , \quad \mathcal{C}^i = 2\sqrt{g} \nabla_j \left(K^{ij} - K h^{ij} \right) \, ,$$

$$K_{ij} = \frac{2\kappa^2}{\sqrt{g}} \left(p_{ij} - \frac{1}{D-2} p h_{ij} \right) \, .$$

For $m = 0$, Hamiltonian vanishes. N, N_i: Lagrange multipliers
$\Rightarrow \mathcal{C} = 0$, $\mathcal{C}_i = 0$: first class constraints $\Leftrightarrow$ general covariance

In $D = 4$,
12 phase space metric components $-$ 4 constraints $-$ 4 gauge symmetries
$=$ 4 phase space degrees of freedom
$=$ degrees of freedom in linearized theory of massless spin 2 graviton

$m \neq 0$ case $\left(h_{ij} \equiv g_{ij} - \delta_{ij}\right)$

$$\eta^{\mu\alpha}\eta^{\mu\beta}\left(h_{\mu\nu}h_{\alpha\beta} - h_{\mu\alpha}h_{\mu\beta}\right)$$
$$= \delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} + 2N_i\left(g^{ij} - \delta^{ij}\right)N_i \,,$$

Action

$$S = \frac{1}{2\kappa^2}\int d^D x \Big\{ p^{ab}\dot{g}_{ab} - N\mathcal{C} - N_i\mathcal{C}^i - \frac{m^2}{4}\Big[\delta^{ik}\delta^{jl}\left(h_{ij}h_{kl} - h_{ik}h_{jl}\right) + 2\delta^{ij}h_{ij} - 2N^2\delta^{ij}h_{ij} + 2N_i\left(g^{ij} - \delta^{ij}\right)N_j\Big]\Big\} \,.$$

N^2, N_iN_j terms $\Rightarrow$ N^2, N_i: *Not* Lagrange multipliers but auxiliary fields.

$$N = \frac{\mathcal{C}}{m^2 \delta^{ij} h_{ij}} , \quad N_i = \frac{1}{m^2} \left(g^{ij} - \delta^{ij} \right)^{-1} \mathcal{C}^j .$$

No constraints nor gauge symmetries.

Hamiltonian:

$$H = \frac{1}{2\kappa^2} \int d^d x \left\{ \frac{1}{2m^2} \frac{\mathcal{C}^2}{\delta^{ij} h_{ij}} + \frac{1}{2m^2} \mathcal{C}^i \left(g^{ij} - \delta^{ij} \right)^{-1} \mathcal{C}^j + \frac{m^2}{4} \left[\delta^{ik} \delta^{jl} \left(h_{ij} h_{kl} - h_{ik} h_{jl} \right) + 2 \delta^{ij} h_{ij} \right] \right\} .$$

12 phase space degrees of freedom, or 6 real degrees of freedom.

One more degree of freedom, compared with linearized theory

$\Rightarrow$ ghost scalar

Boulware-Deser ghost

Massive gravity without ghost

C. de Rham and G. Gabadadze, "Generalization of the Fierz-Pauli Action," Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]],
C. de Rham, G. Gabadadze and A. J. Tolley, "Resummation of Massive Gravity," Phys. Rev. Lett. **106** (2011) 231101 [arXiv:1011.1232 [hep-th]].

S. F. Hassan and R. A. Rosen, "Resolving the Ghost Problem in non-Linear Massive Gravity," Phys. Rev. Lett. **108** (2012) 041101 [arXiv:1106.3344 [hep-th]].

Non-dynamical metric $f_{\mu\nu}$ ($\sim \eta_{\mu\nu}$), $\sqrt{g^{-1}f}$: $\sqrt{g^{-1}f}\sqrt{g^{-1}f} = g^{\mu\lambda}f_{\lambda\nu}$

Minimal extension of Fierz-Pauli action:

$$S = M_p^2 \int d^4x\sqrt{-g}\left[R - 2m^2\left(\mathrm{tr}\sqrt{g^{-1}f} - 3\right)\right] .$$

$\Rightarrow$ vDVZ discontinuity $\Rightarrow$

$$S = M_p^2 \int d^4x \sqrt{-g} \left[R + 2m^2 \sum_{n=0}^{4} \beta_n \, e_n(\sqrt{g^{-1}f}) \right] ,$$

$$e_0(\mathbb{X}) = 1 , \quad e_1(\mathbb{X}) = [\mathbb{X}] , \quad e_2(\mathbb{X}) = \tfrac{1}{2}([\mathbb{X}]^2 - [\mathbb{X}^2]) ,$$

$$e_3(\mathbb{X}) = \tfrac{1}{6}([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]) ,$$

$$e_4(\mathbb{X}) = \tfrac{1}{24}([\mathbb{X}]^4 - 6[\mathbb{X}]^2[\mathbb{X}^2] + 3[\mathbb{X}^2]^2 + 8[\mathbb{X}][\mathbb{X}^3] - 6[\mathbb{X}^4]) ,$$

$$e_k(\mathbb{X}) = 0 \quad \text{for } k > 4 ,$$

$$\mathbb{X} = (X^\mu{}_\nu) , \quad [\mathbb{X}] \equiv X^\mu{}_\mu ,$$

$\sim$ Galileon $\Rightarrow$ Vainshtein mechanism

(longitudinal scalar mode ($h_{\mu\nu} \sim \partial_\mu \partial_\nu \phi$) $\sim$ Galileon scalar field)

For the Galileon scalar π

$$\begin{aligned}
\mathcal{E}_1 =& 1 = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \, , \\
\mathcal{E}_2 =& -2\mathrm{tr}\,\Pi = -\frac{1}{3}\epsilon_{\mu\nu\rho}{}^{\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\delta\partial_\sigma\pi \, , \\
\mathcal{E}_3 =& -3\left\{(\mathrm{tr}\,\Pi)^2 - \mathrm{tr}\,\Pi^2\right\} = -\frac{3}{2}\epsilon_{\mu\nu}{}^{\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\gamma\partial_\rho\pi\partial_\delta\partial_\sigma\pi \, , \\
\mathcal{E}_4 =& -2\left\{(\mathrm{tr}\,\Pi)^3 - 3\mathrm{tr}\,\Pi\mathrm{tr}\,\Pi^2 + 2\mathrm{tr}\,\Pi^3\right\} \\
=& -2\epsilon_{\mu}{}^{\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\beta\partial_\nu\pi\partial_\gamma\partial_\rho\pi\partial_\delta\partial_\sigma\pi \, , \\
\mathcal{E}_5 =& -\frac{5}{6}\left\{(\mathrm{tr}\,\Pi)^4 - 6\left(\mathrm{tr}\,\Pi\right)^2\mathrm{tr}\,\Pi^2 + 8\left(\mathrm{tr}\,\Pi\right)\mathrm{tr}\,\Pi^3 + 3\left(\mathrm{tr}\,\Pi^2\right)^2 - 6\mathrm{tr}\,\Pi^4\right\} \\
=& -\frac{5}{6}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}\partial_\alpha\partial_\mu\pi\partial_\beta\partial_\nu\pi\partial_\gamma\partial_\rho\pi\partial_\delta\partial_\sigma\pi \, .
\end{aligned}$$

Here $\Pi^\mu{}_\nu \equiv \partial^\mu\partial_\nu\pi$ and $\mathrm{tr}\,\Pi = \Pi^\mu{}_\mu$.

Formula

$$
\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma} =& \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\gamma\rho}\eta^{\delta\sigma} - \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\gamma\sigma}\eta^{\delta\rho} + \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\gamma\sigma}\eta^{\delta\nu} \\
&- \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\gamma\nu}\eta^{\delta\sigma} + \eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\gamma\nu}\eta^{\delta\rho} - \eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\gamma\rho}\eta^{\delta\nu} \\
&- \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\gamma\sigma}\eta^{\delta\mu} + \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\gamma\mu}\eta^{\delta\sigma} - \eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\gamma\mu}\eta^{\delta\rho} \\
&+ \eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\gamma\rho}\eta^{\delta\mu} - \eta^{\alpha\nu}\eta^{\beta\mu}\eta^{\gamma\rho}\eta^{\delta\sigma} + \eta^{\alpha\nu}\eta^{\beta\mu}\eta^{\gamma\sigma}\eta^{\delta\rho} \\
&+ \eta^{\alpha\rho}\eta^{\beta\sigma}\eta^{\gamma\mu}\eta^{\delta\nu} - \eta^{\alpha\rho}\eta^{\beta\sigma}\eta^{\gamma\nu}\eta^{\delta\mu} + \eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\gamma\nu}\eta^{\delta\sigma} \\
&- \eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\gamma\sigma}\eta^{\delta\nu} + \eta^{\alpha\rho}\eta^{\beta\nu}\eta^{\gamma\sigma}\eta^{\delta\mu} - \eta^{\alpha\rho}\eta^{\beta\nu}\eta^{\gamma\mu}\eta^{\delta\sigma} \\
&- \eta^{\alpha\sigma}\eta^{\beta\mu}\eta^{\gamma\nu}\eta^{\delta\rho} + \eta^{\alpha\sigma}\eta^{\beta\mu}\eta^{\gamma\rho}\eta^{\delta\nu} - \eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\gamma\rho}\eta^{\delta\mu} \\
&+ \eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\gamma\mu}\eta^{\delta\rho} - \eta^{\alpha\sigma}\eta^{\beta\rho}\eta^{\gamma\mu}\eta^{\delta\nu} + \eta^{\alpha\sigma}\eta^{\beta\rho}\eta^{\gamma\nu}\eta^{\delta\mu} \,.
\end{aligned}
$$

Hamiltonian constraint: Minimal extension case

ADM formulation, $f_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$

$$\mathcal{L} = \pi^{ij}\partial_t\gamma_{ij} + NR^0 + N^i R_i - 2m^2\sqrt{\gamma}\, N \left(\mathrm{tr}\sqrt{g^{-1}\eta} - 3\right) .$$

$$(g^{-1}\eta)^{\mu}_{\ \nu} = \frac{1}{N^2}\begin{pmatrix} 1 & N^l\delta_{lj} \\ -N^i & (N^2\gamma^{il} - N^i N^l)\delta_{lj} \end{pmatrix} , \quad N^i = \gamma^{ij} N_j .$$

Highly nonlinear action in $N_\mu \Rightarrow$ New combinations n^i

$$N^i = (\delta^i_j + ND^i{}_j)n^j ,$$

$$D^i{}_j :\ \left(\sqrt{1 - n^T \mathbf{I}\, n}\right) D = \sqrt{(\gamma^{-1} - Dnn^T D^T)\mathbf{I}} ,$$

$$\mathbf{I} = \delta_{ij} , \quad \mathbf{I}^{-1} = \delta^{ij} ,$$

$$\Rightarrow \mathcal{L} = \pi^{ij}\partial_t\gamma_{ij} + NR^0 + R_i(\delta^i_j + N{D^i}_j)n^j$$
$$- 2m^2\sqrt{\gamma}\left[\sqrt{1 - n^T \mathbf{I} n} + N\mathrm{tr}\left(\sqrt{\gamma^{-1}\mathbf{I} - Dnn^T D^T \mathbf{I}}\right) - 3N\right].$$

Linear in N.

$\delta n_i \Rightarrow n^i = -R_j\delta^{ji}\left[4m^4 \det\gamma + R_k\delta^{kl}R_l\right]^{-1/2}$: Not including N.

$$\delta N \Rightarrow R^0 + R_i D^i_j n^j - 2m^2\sqrt{\gamma}\left[\sqrt{1 - n^r\delta_{rs}n^s}\ {D^k}_k - 3\right] = 0\,.$$

+ secondary constraint = 2 constraints.

12 components of γ_{ij} and π^{ij} − 2 constraints

= 10 components (massive spin 2)

Bimetric gravity (bigravity)

S. F. Hassan and R. A. Rosen, "Bimetric Gravity from Ghost-free Massive Gravity," JHEP **1202** (2012) 126 [arXiv:1109.3515 [hep-th]].

Dynamical $f_{\mu\nu}$ (background independent).

$$S = M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n\left(\sqrt{g^{-1}f}\right) ,$$

$$1/M_{\mathrm{eff}}^2 \equiv 1/M_g^2 + 1/M_f^2 .$$

$R^{(g)}$: scalar curvature for $g_{\mu\nu}$, $R^{(f)}$: scalar curvature for $f_{\mu\nu}$.

Spectrum of the linearized theory

Minimal case: $\beta_0 = 3$, $\beta_1 = -1$, $\beta_2 = 0$, $\beta_3 = 0$, $\quad \beta_4 = 1$.

Linearize $\quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_g} h_{\mu\nu}\,, \quad f_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_f} l_{\mu\nu}\,,$

$$\Rightarrow \ S = \int d^4x \left(h_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} h_{\alpha\beta} + l_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} l_{\alpha\beta} \right)$$
$$- \frac{m^2 M_{\mathrm{eff}}^2}{4} \int d^4x \left[\left(\frac{h^\mu_{\ \nu}}{M_g} - \frac{l^\mu_{\ \nu}}{M_f} \right)^2 - \left(\frac{h^\mu_{\ \mu}}{M_g} - \frac{l^\mu_{\ \mu}}{M_f} \right)^2 \right] .$$

$\hat{\mathcal{E}}^{\mu\nu\alpha\beta}$: usual Einstein-Hilbert kinetic operator.

Change of variables

$$\frac{1}{M_{\mathrm{eff}}} u_{\mu\nu} = \frac{1}{M_f} h_{\mu\nu} + \frac{1}{M_g} l_{\mu\nu} \,, \quad \frac{1}{M_{\mathrm{eff}}} v_{\mu\nu} = \frac{1}{M_g} h_{\mu\nu} - \frac{1}{M_f} l_{\mu\nu} \,.$$

$\Rightarrow$

$$S = \int d^4x \left(u_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} u_{\alpha\beta} + v_{\mu\nu} \hat{\mathcal{E}}^{\mu\nu\alpha\beta} v_{\alpha\beta} \right) - \frac{m^2}{4} \int d^4x \left(v^{\mu\nu} v_{\mu\nu} - v^{\mu}_{\ \mu} v^{\nu}_{\ \nu} \right) \,.$$

One massless spin-2 particle $u_{\mu\nu}$ and one massive spin-2 particle $v_{\mu\nu}$ with mass m.

Dark Energy

Universe can be regarded as isotropic and homogeneous in the scale larger than the clusters of galaxies

$\Rightarrow$ Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \sum_{i,j=1}^{3} \tilde{g}_{ij} dx^i dx^j \,.$$

$a(t)$: scale factor, $\tilde{g}_{ij}$: spacial metric

$\tilde{R}_{ij} = 2K\tilde{g}_{ij}$ ($\tilde{R}_{ij}$: Ricci curvature given by $\tilde{g}_{ij}$)

$K > 0$: sphere, $K < 0$: hyperboloid, $K = 0$: flat space

$$\left\{ \begin{array}{ll} da(t)/dt > 0 & : \text{expanding universe} \\ d^2a(t)/dt^2 > 0 & : \text{accelerating expansion} \end{array} \right.$$

Assume the Universe is filled with perfect fluids.
1st FRW equation: (t, t) component of the Einstein eq.

$$0 = -\frac{3}{\kappa^2}H^2 - \frac{3K}{\kappa^2 a^2} + \rho \,, \quad \kappa^2 \equiv 8\pi G$$

2nd FRW equation: (i, j) component

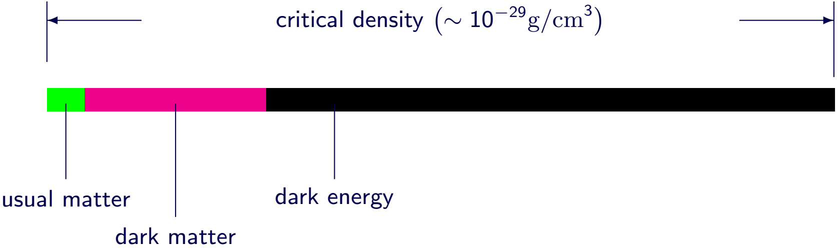
$$0 = \frac{1}{\kappa^2}\left(2\frac{dH}{dt} + 3H^2\right) + \frac{K}{\kappa^2 a^2} + p \,,$$

ρ: energy density, p: pressure, $H \equiv (1/a)\, da(t)/dt$: Hubble rate
The Hubble constant H_0: the present value of H.
$H_0 \sim 70\,\mathrm{km\,s^{-1}Mpc^{-1}} \sim 10^{-33}\,\mathrm{eV}$ in the unit $\hbar = c = 1$.

Cosmic Microwave Background Radiation (CMB) $\Rightarrow K \sim 0$
(When we choose $a(t) = 1$ for the present universe)

$$\rho \sim \rho_c \equiv \frac{3}{\kappa^2} H_0^2 \sim \left(10^{-3}\,\mathrm{eV}\right)^4 \sim 10^{-29} \mathrm{g/cm}^3 \,.$$

ρ_c: critical density. Flat universe $\Rightarrow \rho \sim \rho_c$
Density of usual matter $\sim 4.9\%$, dark matter $\sim 26.8\%$ of ρ_c
$\Longrightarrow$ something unknown $\sim 68.3\% \cdots$ dark energy



Type Ia Supernovae

$\Rightarrow$ accelerating expansion started about 5 billion years ago.

1st and 2nd FRW eqs. $\Rightarrow$ $$\frac{1}{a}\frac{d^2 a(t)}{dt^2} = \frac{dH}{dt} + H^2 = -\frac{\kappa^2}{6}\left(\rho + 3p\right) .$$

accelerating expansion $\Rightarrow p < -\rho/3$

$\Rightarrow$ Dark energy: large negative pressure

Equation of state (EoS) parameter: $w \equiv \frac{p}{\rho}$

Dark energy: $w \sim -1$

Radiation: $w = 1/3$,

Usual matter, cold dark matter (CDM): $w \sim 0$ (dust),

(pressure $\ll$ rest energy)

Cosmological constant: $w = -1$

Dark energy = Cosmological constant??

When EoS parameter w: constant $\Rightarrow$ conservation law:

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0\,,$$

$\Rightarrow \rho = \rho_0 a^{-3(1+w)}$ $(w \neq -1)$, ρ_0: constant of integration

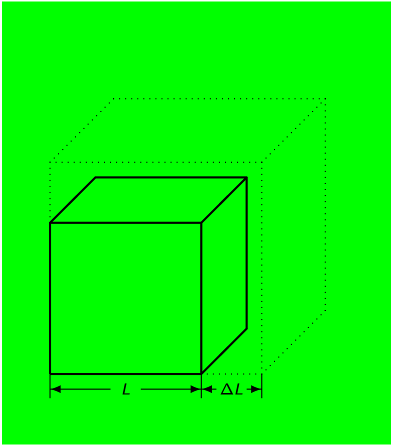
1st FRW eq. $\Rightarrow$

In case $w > -1$, $a(t) \propto t^{\frac{2}{3(1+w)}}$

In case $w < -1$, $a(t) \propto (t_0 - t)^{\frac{2}{3(1+w)}}$
When $t = t_0$, $a(t)$ diverges: Big Rip singularity

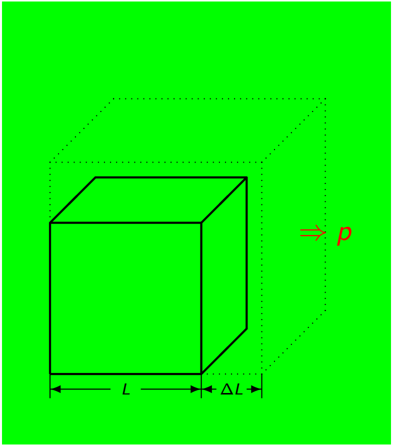
In case $w = -1$, $a(t) \propto a_0 \mathrm{e}^{H_0 t}$, $H_0 \equiv \frac{\rho_0 \kappa^2}{3}$, de Sitter space-time

Cube with sides $L \Rightarrow$ Each side becomes longer by ΔL



Negative Pressure

Internal energy U changes due to pressure p.



$$\Delta U \sim -p\Delta V \sim -3pL^2\Delta L$$

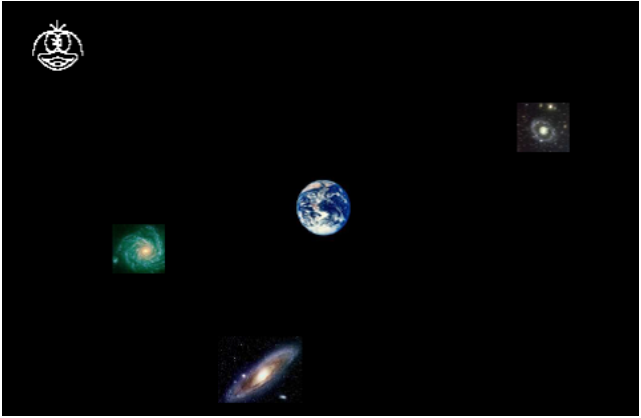
If the pressure is negative, the internal energy increases by the expansion!

The total energy is *NOT* conserved if we take into account for gravity!

If there are phantoms in the universe, the density of the phantom increases by the expansion of the universe.

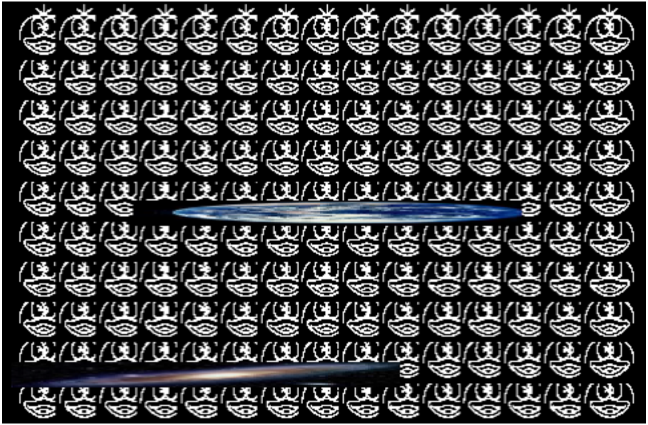
$$H = \frac{\dot{a}}{a} \propto \sqrt{\rho}$$

If the density of the phantoms increases, the ratio H of the expansion also increases.



If there is a phantom,

Phantoms grow up $\Rightarrow$ H increases



The universe is fulfilled by phantoms.


Finally the universe is ripped!

Fine-tuning problem and/or Coincidence problem

Fine-tuning problem, Coincidence problem:

The definitions slightly depend on persons.

A. 1st and 2nd FRW equations ($K = 0$)

$$0 = -\frac{3}{\kappa^2} H^2 + \frac{\Lambda}{2\kappa^2} + \rho_{\mathrm{matter}} \, , \quad 0 = \frac{1}{\kappa^2}\left(2\frac{dH}{dt} + 3H^2\right) - \frac{\Lambda}{2\kappa^2} + p_{\mathrm{matter}} \, ,$$

Λ: cosmological constant

If the dark energy comes from the cosmological term, the cosmological constant is unnaturally small.

$$\sqrt{\Lambda} \sim 10^{-33}\,\mathrm{eV} \ll M_{\mathrm{Planck}} \sim 1/\kappa \sim 10^{19}\,\mathrm{GeV} = 10^{28}\,\mathrm{eV}$$

Strange, especially from the viewpoint of the particle physics.

B. Anthropic principle?

$\frac{\Lambda}{2\kappa^2} \sim \rho_{\rm matter}$ (including dark matter)   Very accidental! if Λ is a constant

Age of the Universe: 13.7 billion years

$$\sim \left(10^{-33}\,{\rm eV}\right)^{-1} \sim \Lambda^{-\frac{1}{4}}$$

Present temperature of the Universe: $(3K)$

$$\sim 10^{-3}\,{\rm eV} \sim \left(\rho_{\rm matter}\right)^{1/4} \sim \left(\frac{\Lambda}{2\kappa^2}\right)^{1/4}$$

$\Rightarrow$ Dark energy might be dynamical?

C. Initial condition?

If the dark energy is a perfect fluid whose EoS parameter $w \sim -1$,

$$\rho_{\mathrm{DE}} = \rho_{\mathrm{DE}\,0} a^{-3(1+w)} \sim \rho_{\mathrm{DE}\,0}$$

Usual matter or CDM (dust with $w = 0$)

$$\rho_{\mathrm{matter}} = \rho_{\mathrm{matter}\,0} a^{-3}$$

Ratio of densities of the dark energy to usual matter and dark matter

$$\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim \left(\rho_{\mathrm{DE}\,0}/\rho_{\mathrm{matter}\,0}\right) a^{-3}$$

In order that $\rho_{\mathrm{DE}\,0} \sim \rho_{\mathrm{matter}\,0}$ in the present Universe,
because the ratio is given by $\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim a^{-3}$,
when transparent to radiation ($a \sim 10^{-3}$), for example:

$$\rho_{\mathrm{DE}}/\rho_{\mathrm{matter}} \sim 10^{-9}$$

We need to fine-tune the initial condition of the ratio.
There might be a model where the dark matter interacts with dark energy and there is a transition between them?
The EoS parameter of the dark energy changes dynamically depending on the expansion (tracker model)?

D. If the dark energy is the vacuum energy,
the quantum corrections from the matter diverge $\sim \Lambda_{\rm cutoff}^4$.

$$\rho_{\rm vacuum} = \frac{1}{(2\pi)^3}\int d^3k \frac{1}{2}\sqrt{k^2+m^2} \sim \Lambda_{\rm cutoff}^4$$

$\Lambda_{\rm cutoff}$: cutoff scale
If the supersymmetry is restored in the high energy,
the vacuum energy by the quantum corrections $\sim \Lambda_{\rm cutoff}^2 \Lambda_{\not{\rm SUSY}}^2$

$$\begin{aligned}\rho_{\rm vacuum} =& \frac{1}{(2\pi)^3}\int d^3k \frac{1}{2}\left(\sqrt{k^2+m_{\rm boson}^2} - \sqrt{k^2+m_{\rm fermion}^2}\right) \\ \sim& \Lambda_{\rm cutoff}^2 \Lambda_{\not{\rm SUSY}}^2\end{aligned}$$

$\Lambda_{\not{\rm SUSY}}$: the scale of the supersymmetry breaking.
$\Lambda_{\not{\rm SUSY}}^2 = m_{\rm boson}^2 - m_{\rm fermion}^2$.

If we use the counter term in order to obtain the very small vacuum energy $\left(10^{-3}\,\mathrm{eV}\right)^4$, we need very very fine-tuning and extremely unnatural.

Maybe we do not understand quantum gravity?

We will discuss later.

The above problems could be clues to understand the gravity.

Modified gravities

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}$$

If the dark energy is a perfect fluid filling the Universe
$\cdots$ modification of the energy momentum tensor $T_{\mu\nu}$ of matters
(r.h.s. in the Einstein equation).
Many models to consider the modification of the Einstein tensor
(l.h.s.)
$\cdots$ Modified gravity models

$F(R)$ gravity, scalar-tensor theory (Brans-Dicke type model),
Gauss-Bonnet gravity, $F(G)$ gravity, massive gravity, bigravity ...

Scalar-tensor theory

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\}, \quad V(\phi) \sim \phi^{-n}: \text{ tracker}.$$

Brans-Dicke type ($\Leftarrow$ higher dimensional theory?)

$$S = \int d^4x \sqrt{-g} \left\{ f(\phi) R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\}.$$

k-essence (X-matter)

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - F\left(\partial_\mu \phi \partial^\mu \phi\right) \right\},$$

$$F\left(\partial_\mu \phi \partial^\mu \phi\right) \rightarrow F\left(\partial_\mu \phi \partial^\mu \phi, \phi\right): \text{ generalized } k\text{-essence}.$$

Scalar-Gauss-Bonnet gravity ($\Leftarrow \alpha'$ correction in string theory?)

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - f(\phi)G\right\} ,$$
$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} .$$

$F(R)$ gravity
(low energy effective theory after integrating heavy fields?)

$$S = \int d^4x\sqrt{-g}F(R) .$$

$F(G)$ gravity

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R + F(G)\right\} .$$

Galileon model, Born-Infeld gravity, massive gravity, bigravity...
Recently there have been remarkable progresses in the study of massive gravity and bigravity.

$F(R)$ bigravity

Standard $F(R)$ gravity $\Leftrightarrow$ scalar tensor theory

$$S_{F(R)} = \int d^4x\sqrt{-g}\left(\frac{F(R)}{2\kappa^2} + \mathcal{L}_{\rm matter}\right) .$$

Introducing the auxiliary field A,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R - A\right) + F(A)\right\} .$$

Variation of $A \Rightarrow A = R$: original action

Rescaling of metric

$$g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}\,, \quad \sigma = -\ln F'(A)\,.$$

$\Rightarrow$ Einstein frame action (Einstein-Hilbert action + real scalar field σ):

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right) ,$$
$$V(\sigma) = \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} F\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}\,.$$

$A = g\left(\mathrm{e}^{-\sigma}\right) \Leftarrow \sigma = -\ln F'(A)$
Coupling of σ with matters appears by the rescaling $g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}$.

Construction of $F(R)$ bigravity
(by the inverse process from scalar-tensor form)
Adding the following actions to the bigravity action

$$S_\varphi = - M_g^2 \int d^4x \sqrt{-\det g} \left\{ \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \right\} + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(\mathrm{e}^\varphi g_{\mu\nu}, \Phi_i \right) ,$$

$$S_\xi = - M_f^2 \int d^4x \sqrt{-\det f} \left\{ \frac{3}{2} f^{\mu\nu} \partial_\mu \xi \partial_\nu \xi + U(\xi) \right\} .$$

Scale transformations $g_{\mu\nu} \to \mathrm{e}^{-\varphi} g_{\mu\nu}$, $f_{\mu\nu} \to \mathrm{e}^{-\xi} f_{\mu\nu}$,

$$\begin{aligned} S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}} \left\{ \mathrm{e}^{-\xi} R^{\mathrm{J}(f)} - \mathrm{e}^{-2\xi} U(\xi) \right\} \\ & + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}} \right) \\ & + M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \left\{ \mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} - \mathrm{e}^{-2\varphi} V(\varphi) \right\} \\ & + \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i \right) . \end{aligned}$$

Kinetic terms of φ and ξ vanish. ($\mathcal{O}^J$: quantities in the Jordan fame)
Coupling of φ with matters also disappears.

Variation of φ and $\xi \Rightarrow$

$$0 = 2m^2 M_{\rm eff}^2 \sum_{n=0}^{4} \beta_n \left(\frac{n}{2} - 2\right) \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{{g^{\rm J}}^{-1} f^{\rm J}}\right) + M_g^2 \left\{ -\mathrm{e}^{-\varphi} R^{{\rm J}(g)} + 2\mathrm{e}^{-2\varphi} V(\varphi) + \mathrm{e}^{-2\varphi} V'(\varphi) \right\} ,$$

$$0 = -\, 2m^2 M_{\rm eff}^2 \sum_{n=0}^{4} \frac{\beta_n n}{2} \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n \left(\sqrt{{g^{\rm J}}^{-1} f^{\rm J}}\right) + M_f^2 \left\{ -\mathrm{e}^{-\xi} R^{{\rm J}(f)} + 2\mathrm{e}^{-2\xi} U(\xi) + \mathrm{e}^{-2\xi} U'(\xi) \right\} .$$

In principle, can be solved algebraically with respect to φ and ξ

$$\varphi = \varphi \left(R^{(g)}, R^{(f)}, e_n \left(\sqrt{g^{-1} f} \right) \right) , \quad \xi = \xi \left(R^{(g)}, R^{(f)}, e_n \left(\sqrt{g^{-1} f} \right) \right) .$$

$\Rightarrow$ analogue of $F(R)$ gravity:

$$S_F = M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}} F^{(f)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right)$$
$$+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi\left(R^{\mathrm{J}(g)}, e_n\left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}}\right)\right)} e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right)$$
$$+ M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} F^{\mathrm{J}(g)} \left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n \left(\sqrt{{g^{\mathrm{J}}}^{-1} f^{\mathrm{J}}} \right) \right)$$
$$+ \int d^4x \mathcal{L}_{\mathrm{matter}} \left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i \right) ,$$

Here

$$F^{\mathrm{J}(g)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(g)} \right.$$
$$\left. - \mathrm{e}^{-2\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} V\left(\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)\right) \right\},$$
$$F^{(f)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(f)} \right.$$
$$\left. - \mathrm{e}^{-2\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} U\left(\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)\right) \right\}.$$

It is difficult to explicitly solve equations with respect to φ and ξ and it might be better to define the model by introducing the auxiliary scalar fields φ and ξ.

Usually we start from a given model and investigate the development of the universe etc. by using the given equations. Here we consider the inverse, that is, for a given development of the universe, we construct a model which reproduces the development, which we call reconstruction.

Minimal case:

$$S_{\mathrm{bi}} = M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \left(3 - \mathrm{tr}\,\sqrt{g^{-1}f} + \det \sqrt{g^{-1}f}\right) .$$

Start from the Einstein frame action. Neglect matter.

$\delta g_{\mu\nu} \ \Rightarrow$

$$\begin{aligned} 0 =& M_g^2 \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu} \right) \\ &+ m^2 M_{\mathrm{eff}}^2 \left\{ g_{\mu\nu} \left(3 - \mathrm{tr}\, \sqrt{g^{-1} f} \right) + \frac{1}{2} f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\qquad\nu} + \frac{1}{2} f_{\nu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}_{\qquad\mu} \right\} \\ &+ M_g^2 \left[\frac{1}{2} \left(\frac{3}{2} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + V(\varphi) \right) g_{\mu\nu} - \frac{3}{2} \partial_\mu \varphi \partial_\nu \varphi \right] . \end{aligned}$$

$\delta f_{\mu\nu} \ \Rightarrow$

$$\begin{aligned} 0 =& M_f^2 \left(\frac{1}{2} f_{\mu\nu} R^{(f)} - R^{(f)}_{\mu\nu} \right) \\ &+ m^2 M_{\mathrm{eff}}^2 \sqrt{\det \left(f^{-1} g \right)} \left\{ - \frac{1}{2} f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{\rho}_{\ \nu} - \frac{1}{2} f_{\nu\rho} \left(\sqrt{g^{-1} f} \right)^{\rho}_{\ \mu} + \det \left(\sqrt{g^{-1} f} \right) f_{\mu\nu} \right\} \\ &+ M_f^2 \left[\frac{1}{2} \left(\frac{3}{2} f^{\rho\sigma} \partial_\rho \xi \partial_\sigma \xi + U(\xi) \right) f_{\mu\nu} - \frac{3}{2} \partial_\mu \xi \partial_\nu \xi \right] . \end{aligned}$$

$\delta\varphi,\ \delta\xi \Rightarrow$

$$0 = -3\Box_g \varphi + V'(\varphi)\,, \quad 0 = -3\Box_f \xi + U'(\xi)\,.$$

$\Box_g,\ \Box_f$: d'Alembertian w.r.t. g, f.

Bianchi identity $0 = \nabla_g^\mu \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu} \right)$ + field equations $\Rightarrow$

$$0 = - g_{\mu\nu} \nabla_g^\mu \left(\mathrm{tr}\, \sqrt{g^{-1} f} \right) + \frac{1}{2} \nabla_g^\mu \left\{ f_{\mu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}{}_{\nu} + f_{\nu\rho} \left(\sqrt{g^{-1} f} \right)^{-1\,\rho}{}_{\mu} \right\} .$$

Similarly

$$0 = \nabla_f^\mu \left[\sqrt{\det\left(f^{-1} g\right)} \left\{ -\frac{1}{2} \left(\sqrt{g^{-1} f} \right)^{-1\nu}{}_{\sigma} g^{\sigma\mu} - \frac{1}{2} \left(\sqrt{g^{-1} f} \right)^{-1\mu}{}_{\sigma} g^{\sigma\nu} + \det\left(\sqrt{g^{-1} f} \right) f^{\mu\nu} \right\} \right] .$$

In case of the Einstein gravity,
conservation law $\Leftarrow$ Einstein equation $+$ Bianchi identity
or conservation laws $\Leftarrow$ scalar field equations
Scalar field equations are not independent of Einstein equation.

In case of bigravity,
only conservation laws $\Leftarrow$ scalar field equations
Einstein equation $+$ Bianchi identities $+$ scalar field equations
$\Rightarrow$ new equations independent of Einstein equation.
Scalar field equations are independent of Einstein equation.

Assume FRW universes by using the conformal time t

$$ds_g^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = a(t)^2 \left(-dt^2 + \sum_{i=1}^{3} \left(dx^i\right)^2 \right) ,$$

$$ds_f^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu} dx^\mu dx^\nu = -c(t)^2 dt^2 + b(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 .$$

(Most general form assuming the homogeneity, isometry, and flat spacial part.)
$\Rightarrow$

$$\delta g_{tt} \ : \ 0 = -3M_g^2 H^2 - 3m^2 M_{\mathrm{eff}}^2 \left(a^2 - ab\right) + \left(\frac{3}{4}\dot{\varphi}^2 + \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 ,$$

$$\delta g_{ij} \ : \ 0 = M_g^2 \left(2\dot{H} + H^2\right) + m^2 M_{\mathrm{eff}}^2 \left(3a^2 - 2ab - ac\right) + \left(\frac{3}{4}\dot{\varphi}^2 - \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 , \qquad H \equiv \frac{\dot{a}}{a} .$$

H is not exactly Hubble rate $\Leftarrow$ t: conformal time

$$\begin{aligned}
\delta f_{tt} \ : \ 0 =& -3M_f^2 K^2 + m^2 M_{\mathrm{eff}}^2 c^2 \left(1 - \frac{a^3}{b^3}\right) + \left(\frac{3}{4}\dot\xi^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,, \\
\delta f_{ij} \ : \ 0 =& M_f^2 \left(2\dot K + 3K^2 - 2LK\right) + m^2 M_{\mathrm{eff}}^2 \left(\frac{a^3 c}{b^2} - c^2\right) \\
& + \left(\frac{3}{4}\dot\xi^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,. \\
& K \equiv \dot b/b \,, \quad L = \dot c/c \,.
\end{aligned}$$

Both of equations derived from Bianchi identity:

$cH = bK$ or $\frac{c\dot a}{a} = \dot b$.

If $\dot a \neq 0$, we obtain $c = a\dot b/\dot a$.

If $\dot a = 0$, we find $\dot b = 0$, that is, a, b: constant, c can be arbitrary.

Redefinition of the scalar fields: $\varphi = \varphi(\eta)$, $\xi = \xi(\zeta)$.
Identify $\eta = \zeta = t \Rightarrow$

$$\omega(t) M_g^2 = -4M_g^2 \left(\dot{H} - H^2\right) - 2m^2 M_{\rm eff}^2 (ab - ac) ,$$
$$\tilde{V}(t) a(t)^2 M_g^2 = M_g^2 \left(2\dot{H} + 4H^2\right) + m^2 M_{\rm eff}^2 (6a^2 - 5ab - ac) ,$$
$$\sigma(t) M_f^2 = -4M_f^2 \left(\dot{K} - LK\right) - 2m^2 M_{\rm eff}^2 \left(-\frac{c}{b} + 1\right) \frac{a^3 c}{b^2} ,$$
$$\tilde{U}(t) c(t)^2 M_f^2 = M_f^2 \left(2\dot{K} + 6K^2 - 2LK\right) + m^2 M_{\rm eff}^2 \left(\frac{a^3 c}{b^2} - 2c^2 + \frac{a^3 c^2}{b^3}\right) .$$
$$\omega(\eta) = 3\varphi'(\eta)^2 ,\ \tilde{V}(\eta) = V\left(\varphi\left(\eta\right)\right) ,\ \sigma(\zeta) = 3\xi'(\zeta)^2 ,\ \tilde{U}(\zeta) = U\left(\xi\left(\zeta\right)\right) .$$

For arbitrary $a(t)$ and $b(t)$, if we choose $\omega(t)$, $\tilde{V}(t)$, $\sigma(t)$, and $\tilde{U}(t)$ to satisfy the above equations, a model admitting the given $a(t)$ and $b(t)$ evolution can be reconstructed.

Cosmological Models

FRW universe:: $ds^2 = \tilde{a}(t)^2 \left(-dt^2 + \sum_{i=1}^3 \left(dx^i\right)^2\right)$.

$\tilde{a}(t)^2 = \frac{l^2}{t^2}$: de Sitter universe.

$\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$ with $n \neq 1$ case:

Redefinition of time coordinate: $d\tilde{t} = \pm\frac{l^n}{t^n}dt$ $\left(\tilde{t} = \pm\frac{l^n}{n-1}t^{1-n}\right)$

$$\Rightarrow \ ds^2 = -d\tilde{t}^2 + \left(\pm(n-1)\frac{\tilde{t}}{l}\right)^{-\frac{2n}{1-n}} \sum_{i=1}^{3} \left(dx^i\right)^2 .$$

$0 < n < 1$: phantom universe, $n > 1$: quintessence universe,
$n < 0$: decelerating universe

Universe with $a(t) = b(t) = 1$

$a(t) = b(t) = 1$ satisfies the previous constraint.
$\Rightarrow$ Einstein frame metric $g_{\mu\nu}$: flat Minkowski space
Physical metric: the scalar field does not directly coupled with matter.
$\Rightarrow$ the metric we observe: Jordan frame metric $g^{\mathrm{J}}_{\mu\nu} = \mathrm{e}^{\varphi} g_{\mu\nu}$.

$$\begin{aligned}
\omega(t) M_g^2 =& 12 M_g^2 \tilde{H}^2 = 2 m^2 M_{\mathrm{eff}}^2 \left(c - 1 \right) , \\
\tilde{V}(t) M_g^2 =& m^2 M_{\mathrm{eff}}^2 \left(1 - c \right) = -6 M_g^2 \tilde{H}^2 \ \Rightarrow \ c = 1 + \frac{6\tilde{H}^2 M_g^2}{m^2 M_{\mathrm{eff}}^2} , \\
\sigma(t) M_f^2 =& 2 m^2 M_{\mathrm{eff}}^2 \left(c - 1 \right) = 12 M_g^2 \tilde{H}^2 , \\
\tilde{U}(t) M_f^2 =& m^2 M_{\mathrm{eff}}^2 c \left(1 - c \right) = -6 M_g^2 \tilde{H}^2 \left(1 + \frac{6\tilde{H}^2}{m^2 M_{\mathrm{eff}}^2} \right) .
\end{aligned}$$

Note: $\omega(t)$, $\sigma(t) > 0$ (no ghost)

Big Rip, quintessence, de Sitter and decelerating universes

$$\tilde{a}(t)^2 = \frac{l^{2n}}{t^{2n}}$$

$$\omega(\eta)^2 M_g^2 = \frac{12 n^2 M_g^2}{\eta^2}, \quad \tilde{V}(\eta) M_g^2 = -\frac{6 n^2 M_g^2}{\eta^2},$$

$$\sigma(\zeta) M_f^2 = \frac{12 n^2 M_g^2}{\zeta^2}, \quad \tilde{U}(\zeta) M_f^2 = -\frac{6 n^2 M_g^2}{\zeta^2} \left(1 + \frac{6 n^2}{m^2 M_{\mathrm{eff}}^2 \zeta^2} \right).$$

$$\Rightarrow \quad \mathrm{e}^{\xi} = \frac{n^2}{t^2},$$

$$\left(ds_f^{\mathrm{J}} \right)^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu}^{\mathrm{J}} dx^\mu dx^\nu = \mathrm{e}^{\xi} ds_f^2 = \frac{n^2}{t^2} \left\{ - \left(1 + \frac{6 n^2}{m^2 M_{\mathrm{eff}}^2 t^2} \right)^2 dt^2 + \left(dx^i \right)^2 \right\}.$$

When $t \sim 0$, redefinition:

$$\tilde t \sim \frac{\alpha}{2t^2}\,, \quad \alpha \equiv \frac{6n^3}{m^2 M_{\mathrm{eff}}^2 t^2}\,,$$

$\Rightarrow$

$$\left(ds_f^{\mathrm{J}}\right)^2 \sim -d\tilde t^2 + \frac{2n^2 \tilde t}{\alpha}\left(dx^i\right)^2 \,.$$

$t \to 0$ (Big Bang or Big Rip) $\Leftrightarrow \tilde t \to +\infty$.
There does not occur singularity in the metric $\left(ds_f^{\mathrm{J}}\right)^2$ because the scale factor $\tilde a$ which is proportional to $\tilde t$ corresponds to the universe filled with radiation.

Super-luminal mode in bigravity

There can be a signal whose speed is larger than the speed of light.
Speed v_g of the massless particle which propagates in the universe described by $g^{\mathrm{J}}_{\mu\nu}$ or $g_{\mu\nu}$

$$v_g^2 = (dx/dt)^2 = 1 \ \Leftrightarrow \ \text{special relativity.}$$

Speed v_f in $f^{\mathrm{J}}_{\mu\nu}$ or $f_{\mu\nu}$

$$v_f^2 = (dx/dt)^2 = c(t)^2/b(t)^2$$

If $c(t)/b(t) > 1$, $v_f > 1$ speed of light in g universe.
$c(t) > 1$ except of $\tilde{H} = 0$: $v_f = 1 + \frac{6\tilde{H}^2}{m^2 M_{\mathrm{eff}}^2} > 1$.
v_f is greater than the speed of light. (Causality is not always violated.)

New theory of massive spin-two field

Y. Ohara, S. Akagi and S. Nojiri, "Renormalizable toy model of massive spin two field and new bigravity" Phys. Rev. D **90** (2014) 043006 [arXiv:1402.5737 [hep-th]],
ibid., "Black hole entropy of new bigravity," arXiv:1407.5765 [hep-th]

New ghost free interactions $\cdots$ "pseudo" linear terms

K. Hinterbichler, "Ghost-Free Derivative Interactions for a Massive Graviton," JHEP **1310** (2013) 102 [arXiv:1305.7227 [hep-th]].
(See also, S. Folkerts, A. Pritzel and N. Wintergerst, "On ghosts in theories of self-interacting massive spin-2 particles," arXiv:1107.3157 [hep-th].)

$$
\begin{aligned}
\mathcal{L}_{d,n} \sim& \eta^{\mu_1\nu_1\cdots\mu_n\nu_n}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}\cdots\partial_{\mu_{d-1}}\partial_{\nu_{d-1}}h_{\mu_d\nu_d}h_{\mu_{d+1}\nu_{d+1}}\,,\\
h_{\mu\nu} =& g_{\mu\nu} - \eta_{\mu\nu}\,,\\
\eta^{\mu_1\nu_1\mu_2\nu_2} \equiv& \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}\,,\\
\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \equiv& \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3} - \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_2} + \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_3}\eta^{\mu_3\nu_1}\\
&- \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_3} + \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_1}\eta^{\mu_3\nu_2} - \eta^{\mu_1\nu_3}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_1}\,.
\end{aligned}
$$

- Linear with respect to h_{00} in the Hamiltonian.

$$\eta^{\mu_1\nu_1\mu_2\nu_2} h_{\mu_1\nu_1} h_{\mu_2\nu_2} \sim h_{00} \left(h_{11} + h_{22} + h_{33} \right) + \text{terms not including } h_{00} \,,$$

$$\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \left(\partial_{\mu_1} \partial_{\nu_1} h_{\mu_2\nu_2} \right) h_{\mu_3\nu_3} \sim \left(\partial_1^2 h_{00} \right) \left(h_{22} + h_{33} - 2 h_{23} h_{32} \right) + \cdots \,.$$

- Do not appear the terms which include both of h_{00} and h_{0i}.

Variation of h_{00}

$\Rightarrow$ a constraint for h_{ij} and their conjugate momenta π_{ij}

$+$ secondary constraint

$\Rightarrow$ eliminate the ghost.

Power-counting renormalizable model of the massive spin two particle

$$
\begin{aligned}
\mathcal{L}_{h0} =& \frac{1}{2}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\left(\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}\right)h_{\mu_3\nu_3} - \frac{m^2}{2}\eta^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \\
& - \frac{\mu}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} \\
& - \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \\
=& \frac{1}{2}\left(h\Box h - h^{\mu\nu}\Box h_{\mu\nu} - h\partial^\mu\partial^\nu h_{\mu\nu} - h_{\mu\nu}\partial^\mu\partial^\nu h + 2h_\nu^{\ \rho}\partial^\mu\partial^\nu h_{\mu\rho}\right) \\
& - \frac{m^2}{2}\left(h^2 - h_{\mu\nu}h^{\mu\nu}\right) - \frac{\mu}{3!}\left(h^3 - 3hh_{\mu\nu}h^{\mu\nu} + 2h_\mu^{\ \nu}h_\nu^{\ \rho}h_\rho^{\ \mu}\right) \\
& - \frac{\lambda}{4!}\left(h^4 - 6h^2h_{\mu\nu}h^{\mu\nu} + 8hh_\mu^{\ \nu}h_\nu^{\ \rho}h_\rho^{\ \mu} - 6h_\mu^{\ \nu}h_\nu^{\ \rho}h_\rho^{\ \sigma}h_\sigma^{\ \mu} + 3\left(h_{\mu\nu}h^{\mu\nu}\right)^2\right)
\end{aligned}
$$

m, μ: parameters with the dimension of mass
λ: dimensionless parameters.
$\Rightarrow$ power-counting renormalizable (free from ghost)
Propagator

$$D^m_{\alpha\beta,\rho\sigma} = \frac{1}{2\left(p^2+m^2\right)} \left\{ P^m_{\alpha\rho} P^m_{\beta\sigma} + P^m_{\alpha\sigma} P^m_{\beta\rho} - \frac{2}{D-1} P^m_{\alpha\beta} P^m_{\rho\sigma} \right\} ,$$
$$P^m_{\mu\nu} \equiv \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} .$$

$p^2 \to \infty \Rightarrow D^m_{\alpha\beta,\rho\sigma} \sim \mathcal{O}\left(p^2\right) \cdots$ Not renormalizable

Classical solution

Assume $h_{\mu\nu} = C\eta_{\mu\nu}$, C: constant

$$S = -\int d^4x V(C)\,, \quad V(C) \equiv -6m^2C^2 + 4\mu C^3 + \lambda C^4\,,$$

$C \Leftarrow V'(C) = 0$.

When $\mu = \lambda = 0$, $V(C)$ (Fierz-Pauli model) is unbounded below

$\cdots$ no inconsistency.

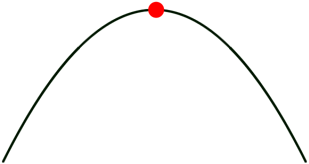
C does not propagate and does not roll down the potential.

(See soon.)

On the other hand,

on the local minimum of the potential ($m^2 < 0$), $h_{\mu\nu}$ becomes tachyon.

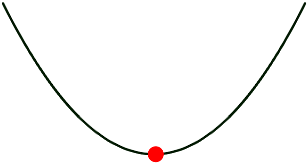
local maximum



no tahyon for $h_{\mu\nu}$ $(m^2 > 0)$

stable

C does not roll down



local minimum

tahyon for $h_{\mu\nu}$ $(m^2 < 0)$!

unstable

We now show C is always a constant.

Eq. of motion:

$$\begin{aligned}0 =&\eta^{\mu\nu\mu_1\nu_1\mu_2\nu_2}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2} - m^2\eta^{\mu\nu\mu_1\nu_1}h_{\mu_1\nu_1} - \frac{\mu}{2}\eta^{\mu\nu\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \\ &- \frac{\lambda}{3!}\eta^{\mu\nu\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}\,.\end{aligned}$$

Assume $h_{\mu\nu} = C\eta_{\mu\nu}$ but C is not a constant,

$$0 = \eta^{\mu\nu}\left(2\Box C - 3m^2 C - 3\mu C^2 + 3\lambda C^3\right) - 2\partial^\mu\partial^\nu C\,.$$

$\Rightarrow$ C should be a constant.

Even if C is on the local maximum of the potential, C does not roll down.

Parametrize m^2 and μ by

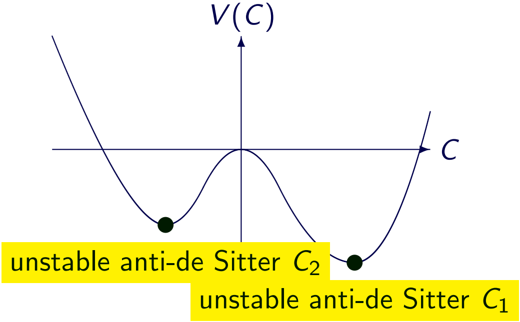
$$m^2 = -\frac{\lambda}{3} C_1 C_2 \, , \quad \mu = -\frac{\lambda}{3} \left(C_1 + C_2 \right) \, .$$

$V'(C) = 0 \Rightarrow C = 0, \ C_1, \ C_2$

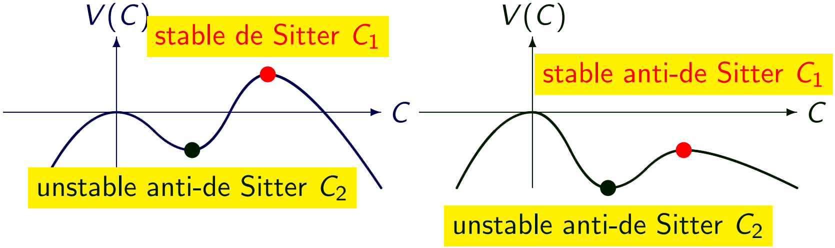
$$V\left(C_1\right) = \frac{\lambda}{3} C_1^3 \left(-C_1 + 2C_2\right) \, , \quad V\left(C_2\right) = \frac{\lambda}{3} C_2^3 \left(-C_2 + 2C_1\right) \, .$$

If we couple the model with gravity, $V(C) \sim$ cosmological constant.
$\Rightarrow$ accelerating expansion of the universe if $V > 0$.

$\lambda > 0$



$\lambda < 0$



When we consider the supersymmetric model, if $E > 0$, the breaking of supersymmetry?

The relation between $C_{1,2}$ and the corresponding space-time and the stability of the solutions

	$0 < \lambda$	$-\frac{2\mu^2}{3m^2} < \lambda < 0$	$-\frac{3\mu^2}{4m^2} < \lambda < -\frac{2\mu^2}{3m^2}$
de Sitter	no solution	C_2 (stable)	no solution
Anti-de Sitter	C_1 (unstable) C_2 (unstable)	C_1 (unstable)	C_1 (unstable) C_2 (stable)

$$C_1 = \frac{-3\mu + \sqrt{9\mu^2 + 12m^2\lambda}}{2\lambda}, \quad C_2 = \frac{-3\mu - \sqrt{9\mu^2 + 12m^2\lambda}}{2\lambda}.$$

New bigravity

Couples with gravity $\sim$ new bigravity (gravity is not renormalizable)

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} \right. \\ -\frac{1}{2}m^2 g^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} - \frac{\mu}{3!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} \\ \left. -\frac{\lambda}{4!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\right\} ,$$

$h_{\mu\nu}$ is not the perturbation in $g_{\mu\nu}$ but $h_{\mu\nu}$ is a field independent of $g_{\mu\nu}$.
Cosmology with the Einstein-Hilbert action:

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R\,.$$

Naively if we work in the local Lorentz frame, no ghost?

I. L. Buchbinder, D. M. Gitman, V. A. Krykhtin and V. D. Pershin, "Equations of motion for massive spin-2 field coupled to gravity," Nucl. Phys. B **584** (2000) 615 [hep-th/9910188],

I. L. Buchbinder, V. A. Krykhtin and V. D. Pershin, "On consistent equations for massive spin two field coupled to gravity in string theory," Phys. Lett. B **466** (1999) 216 [hep-th/9908028].

Even in case of the Fierz-Pauli model, consistent theory should be

$$S = \int d^D x \sqrt{-g} \left\{ \frac{1}{4} \nabla_\mu h \nabla^\mu h - \frac{1}{4} \nabla_\mu h_{\nu\rho} \nabla^\mu h^{\nu\rho} - \frac{1}{2} \nabla^\mu h_{\mu\nu} \nabla^\nu h \right.$$
$$\left. + \frac{1}{2} \nabla_\mu h_{\nu\rho} \nabla^\rho h^{\nu\mu} + \frac{\xi}{2D} R h_{\mu\nu} h^{\mu\nu} + \frac{1-2\xi}{4D} R h^2 - \frac{m^2}{4} h_{\mu\nu} h^{\mu\nu} + \frac{m^2}{4} h^2 \right\} .$$

Furthermore $R_{\mu\nu} = \frac{1}{D} g_{\mu\nu} R$ or $k^\mu k^\nu R_{\mu\nu} = \frac{1}{D} k^2 R$.

k^μ: time-like Killing vector of the background.

In case of interacting model,

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}\nabla_\mu h\nabla^\mu h - \frac{1}{2}\nabla_\mu h_{\nu\rho}\nabla^\mu h^{\nu\rho} - \nabla^\mu h_{\mu\nu}\nabla^\nu h \right.$$

$$+ \nabla_\mu h_{\nu\rho}\nabla^\rho h^{\nu\mu} + \frac{\xi}{4}Rh_{\alpha\beta}h^{\alpha\beta} + \frac{1-2\xi}{8}Rh^2 + \frac{m^2}{2}g^{\mu_1\nu_1\mu_2\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2}$$

$$- \frac{\mu}{3!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}$$

$$\left. - \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\right\},$$

Changes are only for quadratic terms as in the Fierz-Pauli model.
⇒ We may consider (anti-)de Sitter (-Schwarzschild or Kerr) space-time as exact solutions.

Y. Ohara, S. Akagi and S. Nojiri, "New massive spin two model on a curved spacetime", Phys.Rev. D90 (2014) 12, 123013

Assume $h_{\mu\nu} = C g_{\mu\nu}$, C: constant

$$
\begin{aligned}
S &= - \int d^4 x \sqrt{-g} V(C) + \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} R \,, \\
V(C) &= - \left\{ 6m^2 + (2 - 3\xi) R \right\} C^2 + 4\mu C^3 + \lambda C^4 \,, \\
S &= \left\{ (2 - 3\xi) C^2 + \frac{1}{2\kappa^2} \right\} \int d^4 x \sqrt{-g} \left[R - 2\Lambda_{\mathrm{eff}} \right] \,,
\end{aligned}
$$

Effective mass M: $M^2 \equiv m^2 - 2\mu C - \lambda C^2$,

Effective cosmological constant: $\Lambda_{\mathrm{eff}} \equiv \dfrac{\kappa^2 \left(-6m^2 C^2 + 4\mu C^3 + \lambda C^4 \right)}{2\kappa^2 C^2 (2 - 3\xi) + 1}$.

$\Rightarrow R = 4\Lambda_{\rm eff} \Rightarrow$

$$V_0{}'(C) = 4C\left\{ -2\mu\zeta C^3 + \left(\lambda + 6\zeta m^2\right) C^2 + 3\mu C - 3m^2 \right\} = 0\,.$$

$\zeta \equiv \kappa^2 \left(2 - 3\xi\right)$.
Trivial solution $C = 0$.

$$-2\mu\zeta C^3 + \left(\lambda + 6\zeta m^2\right) C^2 + 3\mu C - 3m^2 = 0\,.$$

$\Rightarrow$

$$C = x + \frac{\lambda + 6\zeta m^2}{6\mu\zeta}\,,$$
$$p = -\frac{1}{3}\left\{ \left(\frac{\lambda + 6\zeta m^2}{2\mu\zeta} \right)^2 + \frac{9}{2\zeta} \right\}\,, \quad q = \frac{2}{27}\left(\frac{\lambda + 6\zeta m^2}{2\mu\zeta} \right)^3 - \frac{\lambda}{4\mu\zeta^2}\,.$$

$\omega \equiv \mathrm{e}^{i2\pi/3} \Rightarrow$ Solution

$$x = \omega^k \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \omega^{3-k} \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} ,$$

$$k = 1, 2, 3 ,$$

Determinant $D = -27q^2 - 4p^3 = -2^2 \cdot 3^3 \left\{ \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 \right\} .$

Except the case $q = p = 0$,

1. $D > 0$ There are three different real solutions.
2. $D < 0$ There is only one real solution.
3. $D = 0$ There are three real solutions but two of them are degenerate with each other.

Stability $\Leftrightarrow$ Higuchi bound

A. Higuchi, "Forbidden Mass Range for Spin-2 Field Theory in De Sitter Space-time," Nucl. Phys. B **282** (1987) 397.

Black hole solution

(ant-) de Sitter-Schwarzschild (Kerr) black hole space-time is an exact solution.

Entropy would not be changed from the case of the Einstein gravity.

c.f. Hassan-Rosen bigravity.case:

T. Katsuragawa and S. Nojiri, "Noether current from surface term, Virasoro algebra and black hole entropy in bigravity," Phys. Rev. D **87** (2013) 10, 104032 [arXiv:1304.3181 [hep-th]],

T. Katsuragawa, "Properties of Bigravity Solutions in a Solvable Class," Phys. Rev. D **89** (2014) 124007 [arXiv:1312.1550 [hep-th]].

Entropy is the sum of the contributions from two metric sectors corresponding to $g_{\mu\nu}$ and $f_{\mu\nu}$.