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The Local Geometry of Carnot Manifolds at Singular Points

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Abstract—In this paper we study the local geometry of Carnot manifolds in a neighborhood of a singular point in the case when horizontal vector fields are $2M$-smooth. Here M is the depth of a Carnot manifold.

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In this paper we study the local geometry of Carnot manifolds (which are also known as sub-Riemannian spaces) in a neighborhood of a singular point for the case of $2M$-smooth horizontal vector fields, where M is the depth of a Carnot manifold. In particular, we establish a local approximation theorem, prove the existence of the tangent cone (in the sense of Gromov) in the Carnot–Carathéodory metric, and study its algebraic and geometric properties.

As is known, in the case of regular points, the tangent cone to a sub-Riemannian space is a Carnot group, i.e., a connected simply connected nilpotent stratified Lie group (see [1, 2] for the case of C^∞-smooth vector spaces and [3–8] for fields of the class C^2). In the case of singular points the tangent cone is a homogeneous space (see [9] for C^∞-smooth vector fields).

It is well-known that the main stages of studying the local geometry consist in the construction of nilpotent approximations and in estimating the divergence of integral lines of the initial vector fields and their nilpotent approximations. The goal of this paper is not only to generalize some methods proposed in [9–12, 1, 2] for the case of vector fields with lesser smoothness, but also to synthesize them with methods described in [3, 4, 6–8]. As distinct from papers [9, 12], when obtaining estimates, we use neither special "privileged " coordinates nor the Newton method. We believe that the development of methods proposed in this paper allows one to decrease further the smoothness for the case of singular points.

The necessity to study the local geometry of sub-Riemannian spaces under the minimal requirements to the smoothness is connected with the solution of optimal control problems [13] and with the theory of sub-elliptic equations [14, 15].

In this paper we study an important particular case of Carnot manifolds, where the smoothness of horizontal vector fields and all their commutators up to the order $M-1$ equals $p := M$.

Definition 1. Let $\mathbb{M}$ be a connected smooth manifold of dimension $\dim \mathbb{M} = N$. We say that a sub-Riemannian structure of the depth M is defined on $\mathbb{M}$, if in a neighborhood U of each point $u \in \mathbb{M}$ a system of the (so-called horizontal) vector fields $\{X_1, \dots, X_m\} \in C^p$, $m \leq N$, is given whose commutators up to the order $M-1$ generate the whole tangent space $T_u\mathbb{M}$ at each point (this condition is called the Hörmander condition).

Putting $H\mathbb{M} = H_1 = \operatorname{span}\{X_1, \dots, X_m\}$ and $H_k = \operatorname{span}\{H_{k-1}, [H_{k-1}, H_1]\}$, we obtain a filtration of the tangent bundle $H\mathbb{M} = H_1 \subseteq H_2 \subseteq \dots \subseteq H_M = T\mathbb{M}$.

Let $I = (i_1, i_2, \dots, i_k)$, $i_j \in \{1, 2, \dots, m\}$, be an arbitrary multi-index of order $|I| = k$. We denote by $X_I = [X_{i_1}, [X_{i_2}, \dots, [X_{i_{k-1}}, X_{i_k}] \dots]$ the corresponding commutator of order $k-1$.

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