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ON THE EINSTEIN EQUATION ON LORENTZIAN MANIFOLDS WITH PARALLEL DISTRIBUTIONS OF ISOTROPIC LINES

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Abstract

Some recent results about Einstein Lorentzian manifolds that admit parallel distributions of isotropic lines are reviewed. We find all holonomy algebras of such manifolds and describe special coordinates that allow us to simplify the Einstein equation. Examples in dimension 4 are considered.

Key words: Lorentzian manifold, Einstein equation, Walker metric, holonomy algebra, recurrent light-like vector field, Petrov classification.

Introduction

Let (M,g) be a Lorentzian manifold admitting a parallel distribution of isotropic lines. On any such manifold (of dimension $n+2$, $n\geq 0$) there exist local coordinates $v,x^1,\dots,x^n,u$, the so-called *Walker coordinates*, such that the metric g has the form

$$g=2\mathrm{d}v\,\mathrm{d}u+h+2A\,\mathrm{d}u+H\cdot(\mathrm{d}u)^2, \tag{1}$$

where $h=h_{ij}(x^1,\dots,x^n,u)\,\mathrm{d}x^i\,\mathrm{d}x^j$ is a u-dependent family of Riemannian metrics, $A=A_i(x^1,\dots,x^n,u)\,\mathrm{d}x^i$ is a u-dependent family of one-forms, and H is a local function on M [1]. The vector field $\partial_v=\dfrac{\partial}{\partial v}$ defines the parallel distribution of isotropic lines. Lorentzian manifolds with this property are of interest both in differential geometry and theoretical physics (e.g. [1–6]). Recently in [5] G.W. Gibbons and C.N. Pope studied the Einstein equation on such Lorentzian manifolds (M,g) and gave some physical interpretation for its solutions.

In Section 1 the Einstein equation for the metric (1) is rewritten as a system of partial differential equations with respect to the components h, A and H defining the metric g. In Section 2 it is shown that the Walker coordinates on an Einstein manifold (M,g) can be chosen in such a way that $A=0$, this gives a simplification of the Einstein equation. In Section 3 we consider examples in dimension 4. In Section 4 all holonomy algebras of the Einstein manifolds (M,g) are given.

1. The form of the Einstein equation

A manifold (M,g) is called *an Einstein manifold* if g satisfies the equation

$$\mathrm{Ric}=\Lambda g,\quad \Lambda\in\mathbb{R},$$

where Ric is the Ricci tensor of the metric g. The number $\Lambda\in\mathbb{R}$ is called *the cosmological constant.* If $\Lambda=0$, i.e. $\mathrm{Ric}=0$, then (M,g) is called *Ricci-flat* or *vacuum Einstein.*

A special example of the metric (1) is the metric of a pp-wave

$$g = 2\mathrm{d}v\,\mathrm{d}u + \sum_{i=1}^{n}(\mathrm{d}x^i)^2 + H\cdot(\mathrm{d}u)^2, \quad \partial_v H = 0. \tag{2}$$

If such metric is Einstein, then it is Ricci-flat, and it is Ricci-flat if and only if $\sum\limits_{i=1}^{n}\partial_i^2 H = 0$.

In [5] it is shown that the Einstein equation for a Lorentzian metric of the form (1) implies

$$H = \Lambda v^2 + vH_1 + H_0, \tag{3}$$

where H_0 and H_1 do not depend on v. Furthermore, in [5] it is proved that Eq. (2) is equivalent to Eq. (3) and the following system of equations:

$$\Delta H_0 - \frac{1}{2}F^{ij}F_{ij} - 2A^i\partial_i H_1 - H_1\nabla^i A_i + 2\Lambda A^i A_i - 2\nabla^i\dot{A}_i + {} + \frac{1}{2}\dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij} + \frac{1}{2}h^{ij}\dot{h}_{ij}H_1 = 0, \tag{4}$$

$$\nabla^j F_{ij} + \partial_i H_1 - 2\Lambda A_i + \nabla^j\dot{h}_{ij} - \partial_i(h^{jk}\dot{h}_{jk}) = 0, \tag{5}$$

$$\Delta H_1 - 2\Lambda\nabla^i A_i + \Lambda h^{ij}\dot{h}_{ij} = 0, \tag{6}$$

$$\mathrm{Ric}_{ij} = \Lambda h_{ij}, \tag{7}$$

where $\Delta H_0 = h^{ij}(\partial_i\partial_j H_0 - \Gamma_{ij}^k\partial_k H_0)$ is the Laplace – Beltrami operator of the metrics $h(u)$ applied to H_0, $F_{ij} = \partial_i A_j - \partial_j A_i$ are the components of the differential of the one-form $A = A_i\mathrm{d}x^i$. A dot denotes the derivative with respect to u.

2. Simplification of the Einstein equation

The Walker coordinates are not defined canonically and any other Walker coordinates $\widetilde{v}, \widetilde{x}^1, \dots, \widetilde{x}^n, \widetilde{u}$ such that $\partial_{\widetilde{v}} = \partial_v$ are given by the following transformation (see [6, 7]):

$$\widetilde{v} = v + f(x^1, \dots, x^n, u), \quad \widetilde{x}^i = \widetilde{x}^i(x^1, \dots, x^n, u), \quad \widetilde{u} = u + c.$$

Using this, in [7] the following theorem is proved:

Theorem 1. *Let (M, g) be a Lorentzian manifold of dimension $n+2$ $(n \geq 2)$ admitting a parallel distribution of isotropic lines. If (M, g) is Einstein with the non-zero cosmological constant Λ, then there exist local coordinates $\big(v, x^1, \dots, x^n, u\big)$ such that the metric g has the form*

$$g = 2\mathrm{d}v\,\mathrm{d}u + h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + (\Lambda v^2 + H_0)(\mathrm{d}u)^2$$

with $\partial_v h_{ij} = \partial_v H_0 = 0$, h_{ij} defines a u-dependent family of Riemannian Einstein metrics with the cosmological constant Λ, satisfying the equations:

$$\Delta H_0 + \frac{1}{2}h^{ij}\ddot{h}_{ij} = 0, \tag{8}$$

$$\nabla^j\dot{h}_{ij} = 0, \tag{9}$$

$$h^{ij}\dot{h}_{ij} = 0, \tag{10}$$

$$\mathrm{Ric}_{ij} = \Lambda h_{ij}, \tag{11}$$

where $\dot{h}_{ij} = \partial_u h_{ij}$. Conversely, any such metric is Einstein.

Thus, we reduce the Einstein equation with $\Lambda \neq 0$ on a Lorentzian manifold with a parallel distribution of isotropic lines to the study of families of Einstein Riemannian metrics satisfying Eqs. (9) and (10).

The required coordinates may be found in the following way. Let g be an Einstein metric written with respect to some coordinates $v, x^1, \dots, x^n, u$ by (1). Since g is an Einstein metric, h satisfies (3). Consider the transformation

$$v \mapsto v + f(x^1, \dots, x^n, u), \quad x^i \mapsto x^i, \quad u \mapsto u.$$

Then H_1 changes to $H_1 - 2\Lambda f$. Taking $f = \frac{1}{2\Lambda} H_1$, we get new coordinates such that $H_1 = 0$. Consider another transformation such that its inverse one has the form

$$v = v, \quad x^i = x^i(\widetilde{x}^1, \dots, \widetilde{x}^n, \widetilde{u}), \quad u = \widetilde{u}. \tag{12}$$

We get

$$\widetilde{A}_i = \frac{\partial x^j}{\partial \widetilde{x}^i} \left(A_j + h_{jk} \frac{\partial x^k}{\partial \widetilde{x}^-} \right).$$

Hence, if the equality

$$\frac{\partial x^i}{\partial \widetilde{u}} = -A_j h^{ji} \tag{13}$$

holds, then $\widetilde{A}_i = 0$. Impose the conditions $x^i(\widetilde{x}^1, \dots, \widetilde{x}^n, \widetilde{u}_0) = \widetilde{x}^i$. Then for each set of numbers $\widetilde{x}^k$ there exists a unique solution $x^i(\widetilde{u})$ of the above system of equations. Since the solution depends smoothly on the initial conditions, we may write the solution in the form $x^i(\widetilde{x}^1, \dots, \widetilde{x}^n, \widetilde{u})$. The obtained functions satisfy Eq. (13). Since $\det \left(\frac{\partial x^i}{\partial \widetilde{x}^j}(\widetilde{u}_0) \right) \neq 0$, we get that $\det \left(\frac{\partial x^i}{\partial \widetilde{x}^j} \right) \neq 0$ for $\widetilde{u}$ near $\widetilde{u}_0$. Under this transformation, $\widetilde{H}_1 = H_1 = 0$. We obtain the required transformation.

Consider now the case $\Lambda = 0$. In [7] the following theorem is proved:

Theorem 2. *Let (M, g) be a Lorentzian manifold with a parallel distribution of isotropic lines and assume that (M, g) is Ricci-flat. Then there exist local coordinates $\big(v, x^1, \dots, x^n, u\big)$ such that the metric is given as*

$$g = 2\mathrm{d}v\, \mathrm{d}u + h_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j + vH_1 (\mathrm{d}u)^2,$$

where H_1 and h_{ij} are smooth functions with $\partial_v h_{kl} = \partial_v H_1 = 0$, satisfying the equations:

$$\frac{1}{2}\dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij} + \frac{1}{2}h^{ij}\dot{h}_{ij}H_1 = 0, \tag{14}$$

$$\partial_i H_1 + \nabla^j \dot{h}_{ij} - \partial_i (h^{jk}\dot{h}_{jk}) = 0, \tag{15}$$

$$\Delta H_1 = 0, \tag{16}$$

$$\mathrm{Ric}_{ij} = 0. \tag{17}$$

Conversely, any such metric is Ricci-flat.

To find the required coordinates it is enough to start with some Walker coordinates $v, x^1, \dots, x^n, u$ and to solve the system of equations (13) for transformation (12).

3. Examples in dimension 4

In [8] it is proved that metric (1) for $n=2$ satisfies the Einstein equation (2) if and only if after a proper choice of coordinates $v,\ x,\ y,\ u$ the metric has the following form:

$$g=\frac{2}{P^2}\,\mathrm{d}z\,\mathrm{d}\bar z+\big(2\mathrm{d}v+2W\mathrm{d}z+2\bar W\,\mathrm{d}\bar z+\big(\Lambda v^2+H_0\big)\,\mathrm{d}u\big)\,\mathrm{d}u, \tag{18}$$

where

$$z=x+iy,\quad 2P^2=|\Lambda|2P_0^2=|\Lambda|\left(1+\frac{\Lambda}{|\Lambda|}z\bar z\right)^2,\quad W=i\partial_z L,$$

the function L is $\mathbb{R}$-valued depending on $z,\ \bar z,\ u$ and satisfying the equation

$$\Delta L=-2\frac{\Lambda}{|\Lambda|}L,$$

where $\Delta=2P_0^2\,\partial_z\,\partial_{\bar z}$ is the Laplace – Beltrami operator of the metric $\frac{2}{P^2}\mathrm{d}z\,\mathrm{d}\bar z$ of the 2-dimensional sphere (or Lobachevsky space). All such functions are given by

$$L=2\mathrm{Re}\left(\phi\partial_z(\ln P_0)-\frac{1}{2}\partial_z\phi\right), \tag{19}$$

where $\phi=\phi(z,u)$ is an arbitrary function holomorphic in z and smooth in u. The function $H_0=H_0(z,\bar z,u)$ can be expressed in a similar way in terms of ϕ and another arbitrary function $\phi_1(z,u)$ holomorphic in z and smooth in u.

In [9] it is shown that the Petrov type of any Einstein metric of the form (1) for $n=2$ is either II or D and it may change from point to point, moreover, it is of type II at generic points.

Example 1 [7]. Let $\phi(z,u)$ be one of $c(u)$, $zc(u)$, $z^2c(u)$; then, using Theorem 1, the above metric can be rewritten as

$$g=\frac{2}{P^2}\,\mathrm{d}z\,\mathrm{d}\bar z+\big(\Lambda v^2+\widetilde H_0\big)(\mathrm{d}u)^2,$$

where $\widetilde H_0$ is a harmonic function, i.e. $\Delta\widetilde H_0=0$.

Although formula (19) gives the complete solution to the Einstein equation, it is not useful for constructing examples of the form obtained in Theorems 1 and 2, since “simple” functions ϕ define complicated functions L and metrics (18). For this reason, in [10] another method of finding partial solutions to the Einstein equation (2) is used. First, the following proposition is proved:

Proposition 1. *Let (M,g) be a Lorentzian manifold of dimension 4 admitting a parallel distribution of isotropic lines. If (M,g) is Einstein with the cosmological constant Λ, then in a neighborhood of each point of M there exist local coordinates v, $x,\ y,\ u$ such that the metric g has one of the following forms:*

1) *if $\Lambda>0$, then*

$$g=2\mathrm{d}v\,\mathrm{d}u+\frac{1}{\Lambda}\Big((\mathrm{d}x)^2+\sin^2x\,(\mathrm{d}y)^2\Big)+\\ +\ 2\left(-\frac{\partial_y f}{\sin x}\mathrm{d}x+\sin x\,\partial_x f\mathrm{d}y\right)\mathrm{d}u+(\Lambda v^2+H_0)(\mathrm{d}u)^2, \tag{20}$$

where H_0 and f are functions depending on x, y, u and satisfying the equations:

$$\Delta_{S^2} f = -2f, \tag{21}$$

$$\Delta_{S^2} H_0 = 4\Lambda f^2 - 2\Lambda\left((\partial_x f)^2 + \frac{1}{\sin^2 x}(\partial_y f)^2\right), \tag{22}$$

where $\Delta_{S^2} = \partial_x^2 + \dfrac{1}{\sin^2 x}\partial_y^2 + \cot x\,\partial_x$ *is the Laplace – Beltrami operator of the sphere metric* $(\mathrm{d}x)^2 + \sin^2 x\,(\mathrm{d}y)^2$*;*

2) *if* $\Lambda < 0$*, then*

$$g = 2\mathrm{d}v\,\mathrm{d}u + \frac{1}{-\Lambda\cdot x^2}\big((\mathrm{d}x)^2 + (\mathrm{d}y)^2\big) + 2(-\partial_y f\mathrm{d}x + \partial_x f\mathrm{d}y)\mathrm{d}u + (\Lambda v^2 + H_0)(\mathrm{d}u)^2, \tag{23}$$

where H_0 and f are functions depending on x, y, u and satisfying the equations:

$$\Delta_{L^2} f = 2f, \tag{24}$$

$$\Delta_{L^2} H_0 = -4\Lambda f^2 - 2\Lambda x^2((\partial_x f)^2 + (\partial_y f)^2), \tag{25}$$

where $\Delta_{L^2} = x^2(\partial_x^2 + \partial_y^2)$ *is the Laplace – Beltrami operator of the metric* $\dfrac{1}{x^2}\big((\mathrm{d}x)^2 + (\mathrm{d}y)^2\big)$ *of the Lobachevsky space* L^2*.*

Conversely, all such metrics are Einstein with the cosmological constant Λ*.*

Partial solutions of Eqs. (21) and (24) can be obtained by finding symmetries of these equations. This can be done using Maple 12. For example, partial solutions of (24) may be found in the following forms: $f(x,y,u) = \psi(x,u)$, $f(x,y,u) = \psi\Big(\dfrac{y}{x}, u\Big)$, $f(x,y,u) = \psi\Big(\dfrac{x^2+y^2}{x}, u\Big)$. Consider several examples from [10].

Example 2. The functions $f = \dfrac{c(u)}{x}$, $c(u)\dfrac{y}{x}$, $c(u)\dfrac{x^2+y^2}{x}$ (where $c(u)$ is a smooth function) are partial solutions of (24). In each case the new coordinates can be chosen in such a way that the metric (23) takes the form:

$$g = 2\mathrm{d}v\,\mathrm{d}u + \frac{1}{-\Lambda\cdot x^2}\big((\mathrm{d}x)^2 + (\mathrm{d}y)^2\big) + (\Lambda v^2 + \widetilde{H}_0)(\mathrm{d}u)^2,$$

where $\widetilde{H}_0$ satisfies $\Delta_{L^2}\widetilde{H}_0 = 0$.

Example 3. The functions $f = x^2 y$ and $H_0 = -\Lambda x^4 y$ are partial solutions of Eqs. (24) and (25). We get the following Einstein metric:

$$\begin{aligned} g = 2\mathrm{d}v\,\mathrm{d}u + \frac{1}{-\Lambda\cdot x^2}\big((\mathrm{d}x)^2 + (\mathrm{d}y)^2\big) &- \\ - \ 2x^2\,\mathrm{d}x\,\mathrm{d}u + 4xy\,\mathrm{d}y\,\mathrm{d}u &+ \big(\Lambda v^2 - \Lambda x^4 y^2\big)(\mathrm{d}u)^2. \end{aligned} \tag{26}$$

The Lie algebra of Killing vector fields is spanned by the vector fields $3v\partial_v + x\partial_x + y\partial_y - 3u\partial_u, \partial_u$.

Consider the transformation with the inverse one given by

$$v = \widetilde{v}, \quad x = \widetilde{x}(1 + 3\Lambda\,\widetilde{u}\,\widetilde{x}^3)^{-1/3}, \quad y = \widetilde{y}(1 + 3\Lambda\,\widetilde{u}\,\widetilde{x}^3)^{2/3}, \quad u = \widetilde{u}.$$

With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}v\,\mathrm{d}u + \frac{1}{-\Lambda}\left(\left(36\Lambda^2x^2y^2u^2 + \frac{1}{x^2(1+3\Lambda ux^3)^2}\right)(\mathrm{d}x)^2 - \right.$$
$$- \; 12\Lambda(1+3\Lambda ux^3)yu\,\mathrm{d}x\,\mathrm{d}y +$$
$$\left. + \frac{(1+3\Lambda ux^3)^2}{x^2} + (\mathrm{d}y)^2\right) + \left(\Lambda v^2 + 3\Lambda x^4y^2 + \frac{x^6}{(1+3\Lambda ux^3)^2}\right)(\mathrm{d}u)^2. \quad (27)$$

The metric g is indecomposable and it is of Petrov type II everywhere.

Example 4. The function $f = c(u)\cos x$ (where $c(u)$ is a smooth function) is a partial solution of (21). In each case the new coordinates can be chosen in such a way that the metric (20) takes the form

$$g = 2\mathrm{d}v\,\mathrm{d}u + \frac{1}{\Lambda}\Big((\mathrm{d}x)^2 + \sin^2 x\,(\mathrm{d}y)^2\Big) + (\Lambda v^2 + \widetilde{H}_0)(\mathrm{d}u)^2,$$

where $\widetilde{H}_0$ satisfies $\Delta_{S^2}\widetilde{H}_0 = 0$.

Example 5. The function $f = \ln\left(\tan\dfrac{x}{2}\right)\cos x + 1$ is a partial solution of (21). We get the following Einstein metric:

$$g = 2\mathrm{d}v\,\mathrm{d}u + \frac{1}{\Lambda}\big((\mathrm{d}x)^2 + \sin^2 x\,(\mathrm{d}y)^2\big) +$$
$$+ \; 2\left(\cos x - \ln\left(\cot\frac{x}{2}\right)\sin^2 x\right)\mathrm{d}y\,\mathrm{d}u + \big(\Lambda v^2 + H_0\big)(\mathrm{d}u)^2, \quad (28)$$

where H_0 is a function satisfying (22). We will find the example of such function below. Consider the transformation

$$\widetilde{v} = v, \quad \widetilde{x} = x, \quad \widetilde{y} = y - \Lambda u\left(\ln\left(\tan\frac{x}{2}\right) - \frac{\cos x}{\sin^2 x}\right), \quad \widetilde{u} = u.$$

With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}v\,\mathrm{d}u + \left(\frac{1}{\Lambda} + \frac{4\Lambda u^2}{\sin^4 x}\right)(\mathrm{d}x)^2 +$$
$$+ \; \frac{4u}{\sin x}\mathrm{d}x\,\mathrm{d}y + \frac{\sin^2 x}{\Lambda}(\mathrm{d}y)^2 + \left(\Lambda v^2 + \widetilde{H}_0\right)(\mathrm{d}u)^2, \quad (29)$$

where $\widetilde{H}_0$ satisfies $\Delta_h\widetilde{H}_0 = -\dfrac{1}{2}h^{ij}\ddot{h}_{ij}$, where h is the Riemannian part of the above metric. An example of such $\widetilde{H}_0$ is $\widetilde{H}_0 = -\Lambda\left(\dfrac{1}{\sin^2 x} + \ln^2\left(\cot\dfrac{x}{2}\right)\right)$. Coming back to the initial coordinates, we get $H_0 = \Lambda\cdot\left(\ln\left(\tan\dfrac{x}{2}\right)\cos x + 1\right)$. The Lie algebra of Killing vector fields of the metric (29) is spanned by the vector fields $\partial_y, \partial_u + \Lambda\left(\dfrac{\cos x}{\sin^2 x} - \ln\left(\tan\dfrac{x}{2}\right)\right)\partial_y$. The metric g is of Petrov type D on the set $\left\{(0,x,y,u)|\ln\left(\cot\dfrac{x}{2}\right)\cos x - 1 = 0\right\}$ and it is of type II on the complement to this set. The metric is indecomposable.

Ricci-flat Walker metrics in dimension 4 are found in [11, 12]. They are of the form

$$g = 2\mathrm{d}v\,\mathrm{d}u + (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + 2A_1\mathrm{d}x\,\mathrm{d}u + (-(\partial_x A_1)v + H_0)(\mathrm{d}u)^2, \quad (30)$$

where A_1 and H_0 satisfy $\partial_v A_1 = \partial_v H_0 = 0$,

$$\partial_x^2 A_1 + \partial_y^2 A_1 = 0, \tag{31}$$

$$\partial_x^2 H_0 + \partial_y^2 H_0 = 2\partial_u \partial_x A_1 - 2A_1 \partial_x^2 A_1 - (\partial_x A_1)^2 + (\partial_y A_1)^2. \tag{32}$$

If $\partial_x A_1 \neq 0$ and the metric g is indecomposable, then g is of Petrov type III at generic points [9, 11–13]. If $\partial_x A_1 = 0$, then this is a pp-wave. If it is indecomposable, then it has Petrov type N at the point where the curvature is non-zero [9, 11–13].

Example 6. It is clear that $A_1 = xy$ and $H_0 = \frac{1}{12}(x^4 - y^4)$ are the solutions of (31) and (32). We get the following Ricci-flat metric:

$$g = 2\mathrm{d}v\,\mathrm{d}u + (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + 2xy\,\mathrm{d}x\,\mathrm{d}u + \left(-yv + \frac{1}{12}(x^4 - y^4)\right)(\mathrm{d}u)^2.$$

The Lie algebra of Killing vector fields of g is spanned by the vector field ∂_u.

Consider the transformation

$$\widetilde{v} = v, \quad \widetilde{x} = xe^{yu}, \quad \widetilde{y} = y, \quad \widetilde{u} = u.$$

With respect to the obtained coordinates, we get

$$g = 2\mathrm{d}v\,\mathrm{d}u + e^{-2yu}(\mathrm{d}x)^2 - 2xue^{-2yu}\,\mathrm{d}x\,\mathrm{d}y + \left(1 + x^2u^2e^{-2yu}\right)(dy)^2 + \\ + \left(-yv - x^2y^2e^{-2yu} - \frac{1}{12}y^4 + \frac{1}{12}x^4e^{-4uy}\right)(\mathrm{d}u)^2. \tag{33}$$

4. Holonomy algebras

Recall that any Riemannian manifold (N, h) can be locally decomposed into a product of a flat space and some Riemannian manifolds that can not be further decomposed [14]. In accordance to this, for the tangent space to (N, h) (that can be identified with $\mathbb{R}^n$, $n = \dim N$) and the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ of (N, h), there exists a decomposition

$$\mathbb{R}^n = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_r} \tag{34}$$

and the corresponding decomposition into the direct sum of ideals

$$\mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r \tag{35}$$

such that each $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ is an irreducible Riemannian holonomy algebra, in particular it coincides with one of the following subalgebras of $\mathfrak{so}(n_i)$: $\mathfrak{so}(n_i)$, $\mathfrak{u}\left(\frac{n_i}{2}\right)$, $\mathfrak{su}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right) \oplus \mathfrak{sp}(1)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right)$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}_7 \subset \mathfrak{so}(8)$ or it is an irreducible symmetric Berger algebra (i.e. it is the holonomy algebra of a symmetric Riemannian manifold and it is different from $\mathfrak{so}(n_i)$, $\mathfrak{u}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right) \oplus \mathfrak{sp}(1)$). It is well known that if the manifold (N, h) is Ricci-flat, then each $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ in the above decomposition is one of $\mathfrak{so}(n_i)$, $\mathfrak{su}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right)$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}_7 \subset \mathfrak{so}(8)$. Conversely, if each $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ is one of $\mathfrak{su}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right)$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}_7 \subset \mathfrak{so}(8)$, then (N, h) is Ricci-flat. Next, if (N, h) is an Einstein manifold with $\Lambda \neq 0$, then each $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of $\mathfrak{so}(n_i)$, $\mathfrak{u}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right) \oplus \mathfrak{sp}(1)$ or with a symmetric Berger algebra, and it holds $n_0 = 0$. Conversely, if $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible and $\mathfrak{h} = \mathfrak{sp}\left(\frac{n_i}{4}\right) \oplus \mathfrak{sp}(1)$ or it is a symmetric Berger algebra, then (N,h) is an Einstein manifold. Thus, Riemannian manifolds with some holonomy algebras are automatically Einstein or Ricci-flat.

In [15] the similar problem is studied for the case of Lorentzian manifolds. Let (M,g) be a Lorentzian manifold with a parallel distribution l of isotropic lines. Without loss of generality we may assume that (M,g) is locally indecomposable; i.e. locally it is not a product of a Lorentzian and of a Riemannian manifold. The tangent space to (M,g) can be identified with the Minkowski space $\mathbb{R}^{1,n+1}$. Let $p, e_1, \dots, e_n, q$ be a Witt basis of $\mathbb{R}^{1,n+1}$ such that $\mathbb{R}p$ corresponds to the distribution l. The holonomy algebra $\mathfrak{g}$ of (M,g) is contained in the maximal subalgebra of $\mathfrak{so}(1,n+1)$ preserving $\mathbb{R}p$,

$$\mathfrak{g} \subset \mathfrak{sim}(n) = \left\{ \left. \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} \right| a \in \mathbb{R},\, X \in \mathbb{R}^n,\, A \in \mathfrak{so}(n) \right\} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.$$

The projection $\mathfrak{h}$ of the holonomy algebra of (M,g) onto $\mathfrak{so}(n)$ has to be a Riemannian holonomy algebra [16]. In [15] the following two theorems are proved:

Theorem 3. *If (M,g) is Ricci-flat, then one of the following holds:*

1. *The holonomy algebra of (M,g) coincides with $(\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n$, and in the decomposition* (35) *of $\mathfrak{h} \subset \mathfrak{so}(n)$ at least one subalgebra $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{u}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right) \oplus \mathfrak{sp}(1)$ or with a symmetric Berger algebra.*
2. *The holonomy algebra of (M,g) coincides with $\mathfrak{h} \ltimes \mathbb{R}^n$, and in the decomposition* (35) *of $\mathfrak{h} \subset \mathfrak{so}(n)$ each subalgebra $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{su}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right)$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$.*

Theorem 4. *If (M,g) is Einstein and not Ricci-flat, then the holonomy algebra of (M,g) coincides with $(\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n$, and in the decomposition* (35) *of $\mathfrak{h} \subset \mathfrak{so}(n)$ each subalgebras $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{u}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right) \oplus \mathfrak{sp}(1)$ or with a symmetric Berger algebra. Moreover, in the decomposition* (34) *it holds $n_{s+1} = 0$.*

In [15] an example of a local Einstein (Ricci-flat) metric with each possible holonomy algebra from the above theorems is constructed.

The above two theorems show that if $n = 2$, i.e. $\dim M = 4$ and (M,g) is Ricci-flat, then either $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{so}(2)) \ltimes \mathbb{R}^2$ or $\mathfrak{g} = \mathbb{R}^2$ (the last case corresponds to pp-waves). If (M,g) is Einstein with $\Lambda \neq 0$, then $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{so}(2)) \ltimes \mathbb{R}^2$. These statements are also proved in [9, 13, 17].

Unlike the case of Riemannian manifolds, it can not be stated that a Lorentzian manifold with some holonomy algebra is automatically an Einstein manifold, but there is a weaker statement. Recall that (M,g) is called *totally Ricci-isotropic* if the image of its Ricci operator is isotropic. If (M,g) is a spin Lorentzian manifold and it admits a parallel spinor, then it is totally Ricci-isotropic (but not necessary Ricci-flat, unlike in the Riemannian case) [3, 4]. In [15] the following theorem is proved:

Theorem 5. *If (M,g) is totally Ricci-isotropic, then its holonomy algebra is the same as in Theorem 3. Conversely, if the holonomy algebra of (M,g) is $\mathfrak{h} \ltimes \mathbb{R}^n$ and in the decomposition* (35) *of $\mathfrak{h} \subset \mathfrak{so}(n)$ each subalgebra $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{su}\left(\frac{n_i}{2}\right)$, $\mathfrak{sp}\left(\frac{n_i}{4}\right)$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$, then (M,g) is totally Ricci-isotropic.*

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Резюме

А.С. Галаев. Уравнения Эйнштейна на лоренцевых многообразиях с параллельным распределением изотропных прямых.

Приведен обзор недавних результатов исследования лоренцевых многообразий Эйнштейна, допускающих параллельные распределения изотропных прямых. Найдены алгебры голономии таких многообразий. Описаны специальные координаты, позволяющие упростить уравнение Эйнштейна. Рассмотрены примеры в размерности 4.

Ключевые слова: лоренцево многообразие, уравнение Эйнштейна, метрика Уокера, алгебра голономии, рекуррентное светоподобное векторное поле, классификация Петрова.

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