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Local Discrepancies in the Problem of Fractional Parts Distribution of a Linear Function

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Abstract—In this paper we consider a problem of distribution of fractional parts of the sequence obtained as multiples of some irrational number with bounded partial quotients of its continued fraction expansion. Local discrepancies are the remainder terms of asymptotic formulas for the number of points of the sequence lying in given intervals. Earlier only intervals with bounded and logarithmic local discrepancies were known. We prove that there exists an infinite set of intervals with arbitrary small growth rate of local discrepancies. The proof is based on the connection of considered problem with some of those from Diophantine approximations.

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It is well-known that for any irrational α, a sequence of fractional parts $\{k\alpha\}$ is uniformly distributed modulo 1. In other words, there is an asymptotic formula $N(\alpha, n, I) \sim n|I|$, where

$$N(\alpha, n, I) = \sharp\{k : 0 \leq k < n,\ \ \{k\alpha\} \in I\}$$

is the number of those elements in sequence $\{k\alpha\}$, $0 \leq k < n$, which lie in some fixed interval $I \subseteq [0; 1)$.

This theorem was independently proved in 1909–1910 by P. Bohl, W. Sierpiński and H. Weyl [1–3], and served as one of the incentives to establish the general theory of uniform distribution [4].

From the point of view of analytic number theory, it is natural to ask the question about residual term $r(\alpha, n, I) = \big|N(\alpha, n, I) - n|I|\big|$ in the asymptotic formula above. The function $r(\alpha, n, I)$ is called a residual term or a local discrepancy of the problem of fractional parts distribution for a linear function.

Often they also consider global discrepancy:

$$\Delta(\alpha, n) = \sup_I r(\alpha, n, I).$$

Recently, there have been plenty of results for global discrepancy $\Delta(\alpha, n)$ including growth rate estimates of right order. The details and bibliography can be found, for instance, in [5–7]. In particular, given an irrationality α with bounded partial quotients to be decomposed as a continued fraction, there is the estimate [6]

$$\Delta(\alpha, n) \leq c(\alpha) \ln n$$

with effectively computable constant $c(\alpha)$.

Local discrepancies $r(\alpha, n, I)$ are less investigated. On one hand, E. Hecke [8] discovered the existence of intervals of bounded remainder with $|I| \in \mathbb{Z} + \alpha\mathbb{Z}$, for which

$$r(\alpha, n, I) = O(1).$$

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